ON THE ELEMENTS OF THE CONTINUED FRACTIONS OF QUADRATIC IRRATIONALS

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Abstract. In this paper, we study the elements of the continued fractions of $\sqrt{Q}$ and $(-1+\sqrt{4Q+1})/2$ $(Q \in \mathbb{N})$. We prove that if the period length of continued fraction of $(-1+\sqrt{4Q+1})/2$ is even, then the middle element is odd (see Theorem 1.4 below), a phenomenon observed first by Arnold [2]. We obtain an analogue theorem for the continued fraction of $\sqrt{Q}$ (see Theorem 1.6 below). We also give the parametrization of positive integers Q such that continued fractions of $\sqrt{Q}$ (respectively, $(-1+\sqrt{1+4Q})/2$) has period of length dividing T, where T is an arbitrary positive integer, which generalize Theorem 3 of Arnold [1]. We explicitly describe the set of positive integers Q such that the continued fraction of $\sqrt{Q}$ has period length equal to 3 or 4.

1. Introduction

This paper is motivated by a series of papers by V. I. Arnold. In [1, 2, 3], by calculating hundreds of examples, Arnold exhibited some interesting statistic results of the continued fractions of quadratic irrationals, though some of them were rediscovered. The aim of this paper is to give the proofs of some results observed first by Arnold.

From Lagrange's theorem we know that the continued fraction of an irrational α is periodic if and only if α is quadratic. In this paper, following Arnold, we focus on the continued fractions of the positive roots of equations $x^2 = Q$ and $x^2 + x = Q$, where Q is a positive rational integer.

First, we introduce some notions of the classical theory of continued fractions (see [4, Chapter IV], [5, 6]). The finite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

is expressed as $[a_0; a_1, \ldots, a_n]$. Considering $a_0, a_1, \ldots, a_n$ as indeterminates, we have

$$[a_0; a_1, \ldots, a_n] = \frac{[a_0, a_1, \ldots, a_n]}{[a_1, \ldots, a_n]},$$

where $[a_0, a_1, \ldots, a_n]$ is a polynomial of $a_0, a_1, \ldots, a_n$. For example, $[a_0, a_1] = a_0a_1+1$. Denote the numerator $[a_0, a_1, \ldots, a_n]$ by A_n. Then the sequence A_n can be calculated recursively: $A_{-1} = 1$, $A_0 = a_0$, $A_{k+1} = a_{k+1}A_k + A_{k-1}$ $(k \geq 0)$. Similarly, denote the denominator $[a_1, \ldots, a_n]$ by B_n. Then $B_{-1} = 0$, $B_0 = 1$, $B_{k+1} = a_{k+1}B_k + B_{k-1}$ $(k \geq 0)$. The symbol $[a_0, a_1, \ldots, a_n]$ can also be computed directly by using Euler's rule [4, p. 72–74]. There is a simple relation between $\{A_n\}$ and $\{B_n\}$:

$$A_nB_{n-1} - B_nA_{n-1} = (-1)^{n-1}. \tag{1.1}$$

Next we list as lemmas some known results which will be used subsequently.

Lemma 1.1. *For all* $a_0, a_1, \ldots, a_n$, $[a_0, a_1, \ldots, a_n] = [a_n, \ldots, a_1, a_0]$.

The proof of this lemma appears at the beginning of Section 2.

Lemma 1.2. *If* $Q \in \mathbb{N}$ *is not a perfect square, then the continued fraction of* $\sqrt{Q}$ *is of the form* $[a_0; \overline{a_1, \ldots, a_n, 2a_0}]$, *where* $a_0 = \lfloor \sqrt{Q} \rfloor, a_n = a_1, a_2 = a_{n-1}, \ldots$. (For the proof, see [5, p. 79], [6, p. 47], or [4, pp. 83-92].)

Lemma 1.3. *Let* Q *be a positive integer such that* $1 + 4Q$ *is not a perfect square. Then the continued fraction of* $(-1 + \sqrt{1 + 4Q})/2$ *is of the form* $[a_0; \overline{a_1, \ldots, a_n, 2a_0 + 1}]$, *where* $a_0 = \lfloor (-1 + \sqrt{1 + 4Q})/2 \rfloor, a_n = a_1, a_2 = a_{n-1}, \ldots$. (For the proof, see [5, p. 105].)

In this paper, we prove the following theorems.

Theorem 1.4. *Let* Q *be a positive integer such that* $1 + 4Q$ *is not a perfect square. If the length of the period of the continued fraction of* $\alpha = (-1 + \sqrt{1 + 4Q})/2$ *is even, then the middle element* a_n *of the continued fraction*

$$\alpha = [a_0; \overline{a_1, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_1, 2a_0 + 1}]$$

is odd.

Remark 1.5. *Arnold* ([2, p. 30]) *described the parities of the 'middle' elements of such continued fractions: "But I have no general proof of this fact, which has been observed in several hundred examples." This theorem gives a proof of his observation.*

Considering the similarity of continued fractions of $\sqrt{Q}$ and $(-1 + \sqrt{1 + 4Q})/2$, we get the following analogue of Theorem 1.4.

Theorem 1.6. *Suppose that the length of the period of the continued fraction of* $\sqrt{Q}$ *is divisible by 4, i.e. the continued fraction is of the form*

$$\sqrt{Q} = [a_0; \overline{a_1, \ldots, a_{2n+1}, a_{2n+2}, a_{2n+1}, \ldots, a_1, 2a_0}].$$

If the elements $a_1, \ldots, a_{2n+1}$ *are all even, then the middle element* a_{2n+2} *is even.*

Arnold ([1], Theorem 3) parameterized those positive integers Q whose square roots $\sqrt{Q}$ have continued fractions of period length $T = 2$. He proved that the continued fraction of the square root of an integer Q has the period of length $T = 2$ if and only if Q belongs to one of the two-parametrical series:

$$\text{(I)} \quad Q = x^2y^2 + x \ (x > 1, y \geq 1),$$
$$\text{(II)} \quad Q = x^2y^2 + 2x \ (x \geq 1, y \geq 1).$$

The following four theorems are generalizations of Arnold's result.

Theorem 1.7. *Let* $Q \in \mathbb{N}$. *Then the continued fraction of* $\sqrt{Q}$ *has period of length* $T = 3$ *if and only if* Q *is of the form*

$$(u + k(4u^2 + 1))^2 + 4ku + 1,$$

where $u, k \in \mathbb{N}$.

Theorem 1.8. *Let* $Q \in \mathbb{N}$. *The continued fraction of* $\sqrt{Q}$ *has period of length* $T = 4$ *if and only if one of the following three conditions holds.*

(I) $Q = (-(2v-1)(2uv-u-v+1)+k(2u-1)(4uv-2u-2v+3))^2+4k(2uv-u-v+1)-(2v-1)^2$, *where* $u, v, k \in \mathbb{N}$ *satisfy* $-(2v-1)(2uv-u-v+1)+k(2u-1)(4uv-2u-2v+3) > 0$;

(II) $Q = (-v(4uv+1) + ku(4uv+2))^2 + k(4uv+1) - 4v^2$, *where* $u, v, k \in \mathbb{N}$ *satisfy* $-v(4uv+1) + ku(4uv+2) > 0$ *and* $-v(4uv+1) + ku(4uv+2) \neq v$;

(III) $Q = (-v(4uv-2v+1) + k(2u-1)(2uv-v+1))^2 + k(4uv-2v+1) - 4v^2$, *where* $u, v, k \in \mathbb{N}$ *satisfy* $-v(4uv-2v+1) + k(2u-1)(2uv-v+1) > 0$ *and* $-v(4uv-2v+1) + k(2u-1)(2uv-v+1) \neq v$.

Generally, fixing a positive integer T, we can get the parametrization of positive integers Q such that the continued fractions of $\sqrt{Q}$ (respectively, $(-1+\sqrt{1+4Q})/2$) have period of length dividing T.

Theorem 1.9. *Let* $Q \in \mathbb{N}$. *Then the continued fraction of* $\sqrt{Q}$ *has period of length dividing* T *if and only if* Q *is of the form*

$$a_0^2 + (-1)^{T+1}[a_2, \ldots, a_2]^2 + k[a_2, \ldots, a_1],$$

where

$$a_0 = \frac{1}{2}\left((-1)^{T+1}[a_2, \ldots, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1]\right)$$

and $a_1, a_2, \ldots \in \mathbb{N}$, $k \in \mathbb{Z}$ *such that*

$$(-1)^{T+1}[a_2, \ldots, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1] \in 2\mathbb{N}.$$

Theorem 1.10. *Let* $Q \in \mathbb{N}$. *Then the continued fraction of* $(-1+\sqrt{1+4Q})/2$ *has period of length dividing* T *if and only if* Q *is of the form*

$$a_0^2 + a_0 + (-1)^{T+1}[a_2, \ldots, a_2]^2 + k[a_2, \ldots, a_1],$$

where

$$a_0 = \frac{1}{2}((-1)^{T+1}[a_2, \ldots, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1] - 1)$$

and $a_1, a_2, \ldots \in \mathbb{N}$, $k \in \mathbb{Z}$ *such that*

$$(-1)^{T+1}[a_2, \ldots, a_2, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1] + 1 \in 2\mathbb{N}.$$

2. Proofs of the Theorems

Proof of Lemma 1.1. From the definition we know that

$$[a_0; a_1, \ldots, a_n, a_{n+1}] = [a_0; [a_1; \ldots, a_n, a_{n+1}]].$$

This means that

$$\frac{[a_0, a_1, \ldots, a_n, a_{n+1}]}{[a_1, \ldots, a_n, a_{n+1}]} = a_0 + \frac{[a_2, \ldots, a_{n+1}]}{[a_1, \ldots, a_{n+1}]} = \frac{a_0[a_1, \ldots, a_{n+1}] + [a_2, \ldots, a_{n+1}]}{[a_1, \ldots, a_{n+1}]}.$$

Thus,

$$[a_0, a_1, \ldots, a_n, a_{n+1}] = a_0[a_1, \ldots, a_n, a_{n+1}] + [a_2, \ldots, a_n, a_{n+1}]. \tag{2.1}$$

Similarly,

$$[a_{n+1}, a_n, \ldots, a_1, a_0] = a_{n+1}[a_n, \ldots, a_1, a_0] + [a_{n-1}, \ldots, a_1, a_0].$$

By induction, this is

$$a_{n+1}[a_0, a_1, \ldots, a_n] + [a_0, a_1, \ldots, a_{n-1}] = a_{n+1}A_n + A_{n-1}.$$

Now, the proof follows from the recursive formula for A_n. □

To prove Theorem 1.4, we need the following lemma.

Lemma 2.1. *Let $a_i \in \mathbb{N}$ $(i = 1, \dots, n)$. If a_n is even, then*

$$[a_1, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_1]$$

is also even.

Proof. We prove the lemma by induction. If $n = 1$, then $[a_1] = a_1$ is even. Suppose that the conclusion holds for all $n \le k$. Then

$$\begin{aligned}
&[a_1, \dots, a_{k+1}, \dots, a_1] && \text{(recursive} \\
&= [a_1, \dots, a_{k+1}, \dots, a_2]a_1 + [a_1, \dots, a_{k+1}, \dots, a_3] && \text{formula of} A_n) \\
&= [a_2, \dots, a_{k+1}, \dots, a_1]a_1 + [a_3, \dots, a_{k+1}, \dots, a_1] && \text{(Lemma 1.1)} \\
&= ([a_2, \dots, a_{k+1}, \dots, a_2]a_1 + [a_2, \dots, a_{k+1}, \dots, a_3])a_1 && \text{(recursive} \\
&\quad +[a_3, \dots, a_{k+1}, \dots, a_2]a_1 + [a_3, \dots, a_{k+1}, \dots, a_3] && \text{formula of} A_n) \\
&= [a_2, \dots, a_{k+1}, \dots, a_2]a_1^2 + 2[a_3, \dots, a_{k+1}, \dots, a_2]a_1 && \\
&\quad +[a_3, \dots, a_{k+1}, \dots, a_3] && \text{(Lemma 1.1)} \\
&\equiv 0 \pmod 2. && \text{(induction assumption)}
\end{aligned}$$

□

The next lemma is needed in the proof of Theorem 1.6. The proof of this lemma is omitted because it is similar to that of Lemma 2.1.

Lemma 2.2. *If $a_1, \dots, a_{2n+1}$ are all even and a_{2n+2} is odd, then*

$$[a_1, \dots, a_{2n+1}, a_{2n+2}, a_{2n+1}, \dots, a_1] \text{ is even, and}$$
$$[a_2, \dots, a_{2n+1}, a_{2n+2}, a_{2n+1}, \dots, a_2] \text{ is odd.}$$

Remark 2.3. $[a_1, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_1]$ *is the sum of certain products formed out of $a_1, \dots, a_{n-1}, a_n$. The products occurring in $[a_1, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_1]$ can be explicitly described by Euler's rule* ([4] p.72–74). They are obtained by omitting several separate pairs of consecutive terms from the whole product $a_1 \cdots a_n \cdots a_1$. Given one way of omitting, you can reverse the order of $a_1, \dots, a_n, \dots, a_1$, then you get a new way of omitting, but the two products are the same. For example, letting $n = 2$, by reversing order, $a_1\cancel{a_2}\cancel{a_1}$ changes to $\cancel{a_1}\cancel{a_2}a_1$. In this way, the only terms of $[a_1, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_1]$ left $\pmod 2$ are the terms symmetric with respect to the middle element a_n. Since $x^2 \equiv x \pmod 2$, we have $[a_1, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_1] \equiv a_n[a_1, \dots, a_{n-1}] \pmod 2$. Using this, one can obtain Lemma 2.1 immediately. For example, letting $n = 4$, the term $\cancel{a_1}\cancel{a_2}a_3\cancel{a_4}\cancel{a_3}a_2a_1$ and the term $a_1a_2\cancel{a_3}\cancel{a_4}\ a_3\cancel{a_2}\cancel{a_1}$ are both equal to $a_1a_2a_3$, so they vanish $\pmod 2$. Thus

$$[a_1, a_2, a_3, a_4, a_3, a_2, a_1] \equiv a_1a_4a_1 + a_3a_4a_3 + a_1a_2a_3a_4a_3a_2a_1 \equiv a_4[a_1, a_2, a_3] \pmod 2.$$

Using the same idea, one can get that

$$[a_1, \dots, a_{n-1}, a_n, a_n, a_{n-1}, \dots, a_1] \equiv (a_n + 1)[a_1, \dots, a_{n-1}] + [a_1, \dots, a_{n-2}] \pmod 2.$$

To simplify the proofs of Theorems 1.4, 1.6, 1.9, and 1.10 we build the following lemma.

Lemma 2.4. *Suppose $\alpha = [a_0; \overline{a_1, \dots, a_1, b}]$. Then*

$$\alpha^2 = -b(\alpha - a_0) - a_0^2 + 2\alpha a_0 + \frac{b[a_2, \dots, a_1] + [a_2, \dots, a_2]}{[a_1, \dots, a_1]}. \tag{2.2}$$

Remark 2.5. *When $T \le 3$, we fix the notations as follows:*

$$\begin{aligned}
T = 1,\quad & [a_2, \dots, a_1] = 0, \quad [a_2, \dots, a_2] = 1, \quad [a_1, \dots, a_1] = 1; \\
T = 2,\quad & [a_2, \dots, a_1] = 1, \quad [a_2, \dots, a_2] = 0, \quad [a_1, \dots, a_1] = a_1; \\
T = 3,\quad & [a_2, \dots, a_1] = a_1, \quad [a_2, \dots, a_2] = 1, \quad [a_1, \dots, a_1] = a_1^2 + 1.
\end{aligned}$$

In these cases, the equations can be verified directly.

Proof of Lemma 2.4. Since $\alpha = [a_0; \overline{a_1, \cdots, a_1, b}]$, we have

$$\begin{aligned}\frac{1}{\alpha - a_0} &= a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_1 + \cfrac{1}{b + (\alpha - a_0)}}} \\ &= \frac{(\alpha - a_0 + b)[a_1, a_2, \ldots, a_2, a_1] + [a_1, a_2, \ldots, a_2]}{(\alpha - a_0 + b)[a_2, \ldots, a_2, a_1] + [a_2, \ldots, a_2]}.\end{aligned}$$

Rearranging the terms and using Lemma 1.1, the equation becomes

$$\alpha^2 = -b(\alpha - a_0) - a_0^2 + 2\alpha a_0 + \frac{b[a_2, \ldots, a_1] + [a_2, \ldots, a_2]}{[a_1, \ldots, a_1]}.$$

□

Proof of Theorem 1.4. We prove the theorem by contradiction. Assume that a_n is even. Using Equation (2.2), we get

$$Q = a_0^2 + a_0 + \frac{(2a_0 + 1)[a_2, \ldots, a_1] + [a_2, \ldots, a_2]}{[a_1, \ldots, a_1]}.$$

From Lemma 2.1 we know that the denominator as well as $[a_2, \ldots, a_2]$ is even. But by Equation (1.1), $[a_2, \ldots, a_2, a_1]$ is relatively prime to it. So, the numerator is odd. This contradicts the fact that $Q \in \mathbb{N}$. The proof is completed. □

Proof of Theorem 1.6. Using Equation (2.2) we get

$$Q = a_0^2 + \frac{2a_0[a_2, \ldots, a_{2n+2}, \ldots, a_2, a_1] + [a_2, \ldots, a_{2n+2}, \ldots, a_2]}{[a_1, \ldots, a_{2n+2}, \ldots, a_1]}.$$

If $a_1, \ldots, a_{2n+1}$ are all even and a_{2n+2} is odd, then from Lemma 2.2, the numerator is odd and the denominator is even. That is a contradiction. □

Proof of Theorem 1.7. If the length of the continued fraction of $\sqrt{Q}$ is 3, then we have

$$\sqrt{Q} = [a_0; a_1, a_1, 2a_0 + (\sqrt{Q} - a_0)],$$

where $a_1 \neq 2a_0$. Calculating recursively we get

$$[a_0, a_1] = a_0 a_1 + 1,\ [a_0, a_1, a_1] = a_0 a_1^2 + a_0 + a_1,\ [a_1, a_1] = a_1^2 + 1.$$

Using Equation (2.2) we obtain

$$Q = a_0^2 + \frac{2a_0 a_1 + 1}{a_1^2 + 1}. \tag{2.3}$$

If a_1 is odd, then the denominator is even and the numerator is odd. This is impossible since $Q \in \mathbb{N}$. So a_1 is even. Write $a_1 = 2u$.

Since $(4u, 4u^2 + 1) = 1$, to make the fraction

$$\frac{2a_0 a_1 + 1}{a_1^2 + 1} = \frac{4ua_0 + 1}{4u^2 + 1}$$

a positive integer, we must have

$$a_0 = u + k(4u^2 + 1),\ k \in \mathbb{N}.$$

(We exclude the case $k = 0$, since then $a_1 = 2a_0$.) Substituting a_0, a_1 with u, k in Equation (2.3), we obtain the expression of Q,

$$Q = (u + k(4u^2 + 1))^2 + 4ku + 1.$$

□

Example 2.6. *The least four* $Q \in \mathbb{N}$ *such that the length of the period of the continued fraction of* $\sqrt{Q}$ *is 3 are* 41, 130, 269, 370. *They correspond to*

$$\begin{array}{rllll} u = & 1, & 1, & 1, & 2; \\ k = & 1, & 2, & 3, & 1. \end{array}$$

Proof of Theorem 1.8. Using Equation (2.2) we have

$$Q = a_0^2 + \frac{2a_0[a_2, a_1] + [a_2]}{[a_1, a_2, a_1]} = a_0^2 + \frac{2a_0(a_1a_2 + 1) + a_2}{a_1(a_1a_2 + 2)}.$$

From Theorem 1.6 we can see the parities of a_1, a_2: both a_1 and a_2 are odd, both are even, or a_1 is odd and a_2 is even. These correspond to the three cases in the theorem. We prove only the second case (the others can be discussed in a similar way). Denote a_1, a_2 as $2u, 2v$, respectively. Then we get

$$Q = a_0^2 + \frac{a_0(4uv + 1) + v}{u(4uv + 2)}.$$

Since $(4uv + 1)^2 - 4uv(4uv + 2) = 1$, the fraction

$$\frac{a_0(4uv + 1) + v}{u(4uv + 2)} = -4v^2$$

when $a_0 = -v(4uv + 1)$. Thus we get all the possible values of a_0 to make the fraction integral $a_0 = -v(4uv + 1) + ku(4uv + 2),\ k \in \mathbb{Z}$. This means

$$Q = (-(4uv + 1)v + ku(4uv + 2))^2 + k(4uv + 1) - 4v^2.$$

The conditions $-v(4uv + 1) + ku(4uv + 2) > 0$ is needed for guaranteeing $a_0 > 0$. In order to guarantee that the period length is 4, we need $a_2 \neq 2a_0$, which is equivalent to the last condition in the statement of the theorem. This completes the proof in case (II). □

Example 2.7.

(I) *Let* $u = v = 1$. *Then* $-(2v - 1)(2uv - u - v + 1) = -1, (2u - 1)(4uv - 2u - 2v + 3) = 3$. *If* $k = 1$, *then* $Q = 7$ *and* $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$. *If* $k = 2$, *then* $Q = 32$ *and* $\sqrt{32} = [5; \overline{1, 1, 1, 10}]$. *Let* $u = 1, v = 2$. *Then* $-(2v - 1)(2uv - u - v + 1) = -6, (2u - 1)(4uv - 2u - 2v + 3) = 5$. *Then* $Q = 23$ *when* $k = 2$ *and* $Q = 96$ *when* $k = 3$. *The continued fractions of* $\sqrt{23}$ *and* $\sqrt{96}$ *are* $[4; \overline{1, 3, 1, 8}]$ *and* $[9; \overline{1, 3, 1, 18}]$, *respectively.*

(II) *Let* $u = v = 1$, *then* $4uv + 1 = 5, u(4uv + 2) = 6$. *If* $k = 1$, *then* $2a_0 = a_1 = a_2$. *That means the period is of length 1. If* $k = 2$, *we get* $Q = 55$ *and* $\sqrt{55} = [7; \overline{2, 2, 2, 14}]$. *Let* $u = 2, v = 1$. *Then* $4uv + 1 = 9, u(4uv + 2) = 20$. *When* $k = 1$, *we have* $Q = 126$ *and* $\sqrt{126} = [11; \overline{4, 2, 4, 22}]$.

(III) *Let* $u = v = 1$. *Then* $-v(4uv - 2v + 1) = -3, (2u - 1)(2uv - v + 1) = 2$. *When* $k = 2$ *we have* $2a_0 = a_2 \neq a_1$. *It is the same to say that the period length is 2. When* $k = 3$ *and* $k = 4$ *we have* $Q = 14$ *and* $Q = 33$, *respectively. The corresponding continued fractions are* $\sqrt{14} = [3; \overline{1, 2, 1, 6}]$ *and* $\sqrt{33} = [5; \overline{1, 2, 1, 10}]$.

Proof of Theorem 1.9. By Lemma 1.2, we can write

$$\sqrt{Q} = [a_0; \overline{a_1, \ldots, a_1, 2a_0}].$$

By Equation (2.2),

$$Q = a_0^2 + \frac{2a_0[a_2, \ldots, a_1] + [a_2, \ldots, a_2]}{[a_1, \ldots, a_1]}.$$

Since $[a_2, \ldots, a_1]^2 - [a_1, \ldots, a_1][a_2, \ldots, a_2] = (-1)^T$, the necessary and sufficient condition for the fraction to be an integer is that

$$2a_0 = (-1)^{T+1}[a_2, \ldots, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1] \quad (k \in \mathbb{Z}). \tag{2.4}$$

Notice that the right side of (2.4) should be a positive even number. Thus we complete the proof. □

Proof of Theorem 1.10. By Lemma 1.3, we can write

$$\frac{-1 + \sqrt{4Q+1}}{2} = [a_0; \overline{a_1, \ldots, a_1, 2a_0 + 1}].$$

Then by Equation (2.2),

$$Q = a_0^2 + a_0 + \frac{(2a_0 + 1)[a_2, \ldots, a_1] + [a_2, \ldots, a_2]}{[a_1, \ldots, a_1]}.$$

As above, the necessary and sufficient condition to make the fraction an integer is

$$2a_0 + 1 = (-1)^{T+1}[a_2, \ldots, a_1][a_2, \ldots, a_2] + k[a_1, \ldots, a_1] \quad (k \in \mathbb{Z}). \tag{2.5}$$

Notice that the right side of (2.5) should be a positive odd number. □

Remark 2.8. *It is easy to see that for any sequence $a_1, \ldots, a_{T-1}$ of natural numbers satisfying $a_1 = a_{T-1}, a_2 = a_{T-2}, \ldots,$ there is a number a_T such that $[\lfloor \frac{a_T}{2} \rfloor; \overline{a_1, \ldots, a_T}]$ is the continued fraction of $\sqrt{Q}$ or $\dfrac{-1 + \sqrt{1+4Q}}{2}$, where Q is a natural number.*

Example 2.9. *Let $T = 6$.*

(I) *Let $a_1 = 1, a_2 = 2, a_3 = 3$. Then $[a_2, a_3, a_2] = 16$, $[a_2, a_3, a_2, a_1] = 23$, $[a_1, a_2, a_3, a_2, a_1] = 33$. If we choose $k = 12$, then $(-1)^{T+1}[a_2, \ldots a_1] + k[a_1 \ldots a_1] = 28$ is even. Thus $2a_0 = 28$, $a_0 = 14, Q = 216$. $\sqrt{216} = [14; \overline{1, 2, 3, 2, 1, 28}]$. If we set $k = 13$, then $(-1)^{T+1}[a_2, \ldots a_1] + k[a_1 \ldots a_1] = 61$ is odd. Thus, $2a_0 + 1 = 61$, $a_0 = 30, Q = 973$. $(-1 + \sqrt{1 + 4 * 973})/2 = [30; \overline{1, 2, 3, 2, 1, 61}]$.*

(II) *Let $a_1 = 1, a_2 = 2, a_3 = 2$. Then $[a_2, a_3, a_2] = 12$, $[a_2, a_3, a_2, a_1] = 17$, $[a_1, a_2, a_3, a_2, a_1] = 24$. Thus $(-1)^{T+1}[a_2, \ldots a_1][a_2, a_3, a_2] + k[a_1 \ldots a_1]$ is always even. In other words, we cannot find an a_0 to make $[a_0; \overline{1, 2, 2, 2, 1, 2a_0 + 1}]$ into the continued fraction of $(-1+\sqrt{1 + 4Q})/2$ for any natural number Q. When we chose $k = 9$, the least integer such that $(-1)^{T+1}[a_2, \ldots a_1][a_2, a_3, a_2] + k[a_1 \ldots a_1] > 0$, we get $a_0 = 6, Q = 45$. The continued fraction is $\sqrt{45} = [6; \overline{1, 2, 2, 2, 1, 12}]$.*

(III) *Let $a_1 = 1, a_2 = 1, a_3 = 3$. Then $[a_2, a_3, a_2] = 5$, $[a_2, a_3, a_2, a_1] = 9$, $[a_1, a_2, a_3, a_2, a_1] = 16$. Thus $(-1)^{T+1}[a_2, \ldots a_1][a_2, a_3, a_2]+k[a_1 \ldots a_1]$ is always odd. So, $[a_0; \overline{1, 1, 3, 1, 1, 2a_0}]$ cannot be the continued fraction of $\sqrt{Q}$ for any natural number Q. When we chose $k = 3$, the least integer such that $(-1)^{T+1}[a_2, \ldots a_1][a_2, a_3, a_2] + k[a_1 \ldots a_1] > 0$, we get $a_0 = 1, Q = 4$. The continued fraction is $(-1 + \sqrt{1 + 4 * 4})/2 = [1; \overline{1, 1, 3}]$. The* *period length is* 3*, a divisor of* 6*. If* $k = 4$*, then* $a_0 = 9, Q = 101$*. The continued fraction is* $(-1 + \sqrt{1 + 4 * 101})/2 = [9; \overline{1, 1, 3, 1, 1, 19}]$.

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