ON THE REPRESENTATION OF CERTAIN REALS VIA THE GOLDEN RATIO

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Abstract. Taking the reciprocal of the golden ratio and summing its non-negative integer powers, we obtain a series that converges. We then consider series obtained by striking out terms of this series, proving key theorems about them and the real numbers to which they converge. Finally, we preassign two-parameter families of real numbers related to the Fibonacci numbers and give their series expansions.

1. Introduction

Consider sequence $\{U_n\}$ defined for all integers n by

$$U_n = U_{n-1} + U_{n-2},\ U_0 = a,\ U_1 = b.$$

When $(a, b) = (0, 1)$, $\{U_n\} = \{F_n\}$, the sequence of Fibonacci numbers. When $(a, b) = (2, 1)$, $\{U_n\} = \{L_n\}$, the sequence of Lucas numbers. The Binet formulas (the closed forms) for F_n and L_n can be found with the use of standard difference techniques. These are defined for all integers n by

$$F_n = \frac{\beta^n - \gamma^n}{\sqrt{5}} \text{ and } L_n = \beta^n + \gamma^n,$$

where

$$\beta = \frac{1+\sqrt{5}}{2} \text{ and } \gamma = \frac{1-\sqrt{5}}{2}.$$

Note that $\beta\gamma = -1$. The real number β is the celebrated golden ratio or golden section.

Set

$$\alpha = \frac{1}{\beta} = -\gamma = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

Relationships that we require and that hold for all integers n are

$$1 = \alpha + \alpha^2, \quad \alpha^n = \alpha^{n+1} + \alpha^{n+2}, \tag{1.1}$$

and

$$\alpha^n = \frac{(-1)^n \left(L_n - F_n\sqrt{5}\right)}{2}. \tag{1.2}$$

Those in (1.1) are immediate, while (1.2) follows with the use of the Binet forms.

Consider

$$\sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \cdots, \tag{1.3}$$

an infinite series of real numbers that converges to $2 + \alpha \approx 2.618$. The focus of this paper is on series, or expansions, obtained from (1.3) by deleting terms. If these expansions are infinite they have the form

$$\sum_{i=0}^{\infty} \alpha^{k(i)} = \alpha^{k(0)} + \alpha^{k(1)} + \alpha^{k(2)} + \cdots, \tag{1.4}$$

in which $\{k(0), k(1), k(2), \ldots\}$ is a strictly increasing infinite sequence of non-negative integers. By the monotone principle of convergence, such series converge. We consider only cases where $\{k(0), k(1), k(2), \ldots\}$ is a proper subsequence of $\{0, 1, 2, \ldots\}$, so that (1.4) converges to a number in the open interval $(0, 2+\alpha)$. We refer to such a series as an *infinite alpha expansion* or simply as an *infinite expansion*.

By a *finite alpha expansion*, or simply a *finite expansion*, we mean a sum

$$\sum_{i=0}^{n} \alpha^{k(i)} = \alpha^{k(0)} + \alpha^{k(1)} + \alpha^{k(2)} + \cdots + \alpha^{k(n)}, \tag{1.5}$$

in which each $k(i)$ is a non-negative integer and $k(0) < k(1) < \cdots < k(n)$.

When specifying an alpha expansion it is sometimes convenient to specify the sequence $\{k(i)\}$ that defines it. For example, we sometimes refer to (1.4) as the infinite alpha expansion defined by (or, whose defining sequence is) $\{k(0), k(1), k(2), \ldots\}$.

We take two alpha expansions to be different if the sequences that define them are different. A finite expansion, therefore, is different from any infinite expansion.

If the infinite expansion (1.4) converges to θ, we say that θ is *represented* by (1.4), or that θ is *represented* by $\{k(0), k(1), k(2), \ldots\}$. Similarly for finite expansions. Of course different expansions can represent the same real number. For example, the three expansions $1+\alpha$, $1+\alpha^2+\alpha^3$, and $1+\alpha^2+\alpha^4+\alpha^6+\cdots$ are all different and all represent $\left(1+\sqrt{5}\right)/2$.

In Section 2 we define a greedy algorithm to obtain alpha expansions and prove certain theoretical results, including existence and uniqueness theorems. In Section 3 we consider the infinite sum $\sum_{i=0}^{\infty} \alpha^{ai+b}$ and establish Fibonacci connections. In Sections 4 and 5 we nominate certain real numbers in the open interval $(0, 2+\alpha)$ and give their alpha expansions. Specifically, these real numbers are two-parameter families derived from the Fibonacci and/or Lucas numbers. In Section 6 we indicate the method of proof for the results in Sections 4 and 5. Finally, in Section 7 we indicate the relationship of the work here with work in older literature.

2. Theoretical Results

Later we prove the central result that every $0 < \theta < 2+\alpha$ has an alpha expansion. For the proof we require the following process.

The Greedy Algorithm. Let θ be a real number $0 < \theta < 2+\alpha$. Choose the least non-negative integer $k(0)$ such that $\theta - \alpha^{k(0)} \geq 0$. If $\theta - \alpha^{k(0)} = 0$, stop. Otherwise, choose the least non-negative integer $k(1) > k(0)$ such that $\left(\theta - \alpha^{k(0)}\right) - \alpha^{k(1)} \geq 0$. If, after n such steps, non-negative integers $k(n-1) > \cdots > k(0)$ are produced such that $\theta - \left(\alpha^{k(0)} + \cdots + \alpha^{k(n-1)}\right) = 0$, the process terminates. Otherwise the process continues indefinitely.

The following lemma, which we require in the sequel, says that in a sequence $\{k(i)\}$, finite or infinite, produced by the greedy algorithm, if two consecutive terms are not consecutive integers, then no two subsequent consecutive terms can be consecutive integers.

Lemma 2.1 (**The Gap Lemma**). *Suppose the greedy algorithm is used on $0 < \theta < 2 + \alpha$ to produce the sequence $\{k(i)\}$, finite or infinite. If $k(i_0 + 1) - k(i_0) \geq 2$ for some i_0, then $k(i+1) - k(i) \geq 2$ for all $i > i_0$.*

Proof. Suppose that $k(i_0+1)-k(i_0) \geq 2$ for some i_0, and that $k(i_0+2)-k(i_0+1) = 1$. Then $\alpha^{k(i_0+1)} + \alpha^{k(i_0+2)} = \alpha^{k(i_0+1)} + \alpha^{k(i_0+1)+1} = \alpha^{k(i_0+1)-1}$. But this says that $\alpha^{k(i_0)}$ and $\alpha^{k(i_0+1)-1}$ are consecutive terms in the expansion, contradicting the hypothesis. This is the first step of an inductive proof whose path is now clear. We leave the details to the reader. $\square$

The theorem that follows settles the question of the existence of alpha expansions.

Theorem 2.2. *Every $0 < \theta < 2 + \alpha$ is represented by an alpha expansion.*

Proof. Apply the greedy algorithm to θ. If the algorithm terminates then θ is represented by the resulting finite alpha expansion.

Next, suppose the greedy algorithm does not terminate and let the sequence that results be $S = \{k(0), k(1), k(2), \ldots\}$. Let n_0 be a non-negative integer for which $k(n_0+1) - k(n_0) \geq 2$. We know that n_0 exists because $0 < \theta < 2 + \alpha$. By the gap lemma $k(n+1) - k(n) \geq 2$ for all $n > n_0$. Let $\varepsilon > 0$ and choose an integer $N > n_0$ such that $N \in S$ and $\varepsilon > \alpha^N$. Such an N exists because S is infinite and $0 < \alpha < 1$. But because $N + 1 \notin S$, $\varepsilon > \alpha^{N+1} > \theta - \left(\alpha^{k(0)} + \cdots + \alpha^N\right) > 0$, demonstrating that $\alpha^{k(0)} + \alpha^{k(1)} + \alpha^{k(2)} + \cdots$ converges to θ. This completes the proof. $\square$

The next theorem indicates precisely which real numbers have an alpha expansion whose defining sequence $k(i)$ does not include any two consecutive integers.

Theorem 2.3. *The real number θ has an alpha expansion whose defining sequence $\{k(i)\}$ does not include any two consecutive integers if and only if $0 < \theta \leq 1 + \alpha$.*

Proof. The infinite alpha expansion $\alpha^0 + \alpha^2 + \alpha^4 + \cdots$ represents $1 + \alpha$, the largest real number whose defining sequence does not contain consecutive integers.

Now suppose that $0 < \theta < 1 + \alpha$. If $\theta = \alpha^{k(0)}$ for some non-negative integer $k(0)$ we are done. Otherwise, with the greedy algorithm, let the first two terms in its expansion be $\alpha^{k(0)} + \alpha^{k(1)}$. If $k(0) = 0$ then $k(1) \geq 2$ (since $0 < \theta < 1 + \alpha$) and we are done by the gap lemma. On the other hand, if $k(0) \geq 1$ we see from (1.1) that $k(1) \geq k(0) + 2$, and the proof is complete by the gap lemma. $\square$

The next theorem addresses the uniqueness of a finite (infinite) expansion where the defining sequence does not include any two consecutive integers.

Theorem 2.4. *No $0 < \theta < 1 + \alpha$ can be represented by different finite (infinite) alpha expansions where the defining sequence of each expansion does not include any two consecutive integers.*

Proof. The proofs of the finite and infinite cases have subtle differences, so we treat them separately. Let $0 < \theta < 1 + \alpha$ be represented by two different finite expansions defined by $\{k(0), \ldots, k(n)\}$ and $\{K(0), \ldots, K(m)\}$, where neither of these expansions includes any two consecutive integers. If $k(i) = K(i)$ for $0 \leq i \leq j$ we can strike out these terms and consider only those that remain. Therefore, without loss of generality, assume that $k(0) < K(0)$.

Then, keeping in mind (1.1), we have

$$\begin{aligned}\alpha^{K(0)}+\cdots+\alpha^{K(m)} &< \alpha^{K(0)}+\alpha^{K(0)+2}+\alpha^{K(0)+4}+\cdots\\ &= \alpha^{K(0)}/\left(1-\alpha^2\right)\\ &= \alpha^{K(0)-1}\\ &\leq \alpha^{k(0)},\end{aligned}$$

a contradiction.

Next, suppose $0<\theta<1+\alpha$ is represented by two different infinite expansions defined by $\{k(0),k(1),k(2),\ldots\}$ and $\{K(0),K(1),K(2),\ldots\}$, where neither of these expansions includes any two consecutive integers. As previously, we may assume that $k(0)<K(0)$. Then

$$\begin{aligned}\alpha^{K(0)}+\alpha^{K(1)}+\alpha^{K(2)}+\cdots &\leq \alpha^{K(0)}+\alpha^{K(0)+2}+\alpha^{K(0)+4}+\cdots\\ &= \alpha^{K(0)-1}\\ &\leq \alpha^{k(0)}\\ &< \alpha^{k(0)}+\alpha^{k(1)}+\alpha^{k(2)}+\cdots,\end{aligned}$$

a contradiction. This proves the theorem. □

In Theorem 2.4 the requirement that both expansions are finite, or that both are infinite, is important. For instance, α^0 and $\alpha+\alpha^3+\alpha^5+\cdots$ are different expansions where one is finite and one is infinite. The defining sequence of each expansion does not include any two consecutive integers, but each of these expansions represents the same real number.

The expansions $1=1$ and $2=1+\alpha+\alpha^2$ represent 1 and 2, respectively. No other rational number in the open interval $(0,2+\alpha)$ is represented by a finite alpha expansion. This is the content of our next theorem.

Theorem 2.5. *Let $\frac{p}{q}\notin\{1,2\}$ be a rational number in the open interval $(0,2+\alpha)$. Then $\frac{p}{q}$ cannot be represented by a finite alpha expansion.*

Proof. Let $\frac{p}{q}\notin\{1,2\}$ be a rational number in the open interval $(0,2+\alpha)$, where p and q are relatively prime integers. Suppose $\frac{p}{q}$ is represented by a finite alpha expansion $G(\alpha)$. It is easy to check that none of the seven finite alpha expansions in which the power of α is at most 2 represent $\frac{p}{q}$. This means that α is a root of the polynomial $qG(x)-p$, and this polynomial has degree 3 or higher. Thus

$$qG(x)-p=\left(x^2+x-1\right)H(x), \tag{2.1}$$

in which $H(x)\in Z[x]$ is a polynomial of degree at least 1. A prime factor, r, of q exists since $\frac{p}{q}$ is not an integer. Let F_r denote the finite field of order r. Over $F_r[x]$, (2.1) becomes $-p=(x^2+x-1)\,H(x)$, a contradiction. Therefore, the assumption that $\frac{p}{q}$ can be represented by a finite alpha expansion is false. □

3. Sums Linked to the Fibonacci and Lucas Numbers

In this section we consider $\sum_{i=0}^{\infty}\alpha^{ai+b}$ under the assumption that a is a positive integer and b is an integer, positive, negative, or zero. First we sum this geometric series and make use of (1.2) to introduce the Fibonacci and Lucas numbers. Then, upon rationalizing the denominator and making use of some standard Fibonacci-Lucas identities, we obtain

$$\sum_{i=0}^{\infty} \alpha^{ai+b} = \frac{\left((-1)^{a+1} L_{a-b} + (-1)^b L_b\right) - \left((-1)^a F_{a-b} + (-1)^b F_b\right)\sqrt{5}}{2(-1)^{a+1}\left(L_a - 1\right) + 2}. \tag{3.1}$$

If a and b are both even (3.1) becomes

$$\sum_{i=0}^{\infty} \alpha^{ai+b} = \frac{L_{a-b} - L_b + \left(F_{a-b} + F_b\right)\sqrt{5}}{2L_a - 4}. \tag{3.2}$$

In (3.2), replacing b by $a-b$ we obtain

$$\sum_{i=0}^{\infty} \alpha^{ai+(a-b)} = \frac{L_b - L_{a-b} + \left(F_{a-b} + F_b\right)\sqrt{5}}{2L_a - 4}. \tag{3.3}$$

Finally, adding (3.2) and (3.3) we obtain

$$\sum_{i=0}^{\infty} \alpha^{ai+b} + \sum_{i=0}^{\infty} \alpha^{ai+(a-b)} = \frac{\left(F_{a-b} + F_b\right)\sqrt{5}}{L_a - 2}. \tag{3.4}$$

Considering all possible parities of a and b, and performing similar manipulations, we have the following theorem.

Theorem 3.1. *Suppose a is a positive integer and b is an arbitrary integer. Then*

$$\sum_{i=0}^{\infty} \alpha^{ai+b} + \sum_{i=0}^{\infty} \alpha^{ai+(a-b)} = \begin{cases} \left(F_{a-b} + F_b\right)\sqrt{5}/\left(L_a - 2\right), & a \text{ even, } b \text{ even;} \\ \left(L_{a-b} + F_b\sqrt{5}\right)/L_a, & a \text{ odd, } b \text{ odd;} \\ \left(L_{a-b} + L_b\right)/\left(L_a - 2\right), & a \text{ even, } b \text{ odd;} \\ \left(L_b + F_{a-b}\sqrt{5}\right)/L_a, & a \text{ odd, } b \text{ even.} \end{cases}$$

Note that in Theorem 3.1 the left side is not necessarily an alpha expansion but can be made so with tighter restrictions on a and b. These restrictions require that no power of α can be negative, and that the sets $\{ai+b\}_{i\geq 0}$ and $\{ai+(a-b)\}_{i\geq 0}$ do not intersect. This leads to the following: a and b must be positive integers with $a > b$ and $a \neq 2b$.

4. Predictable Alpha Expansions I

In this section and the next, we give the alpha expansions of certain two-parameter families of reals associated with the Fibonacci and/or Lucas numbers. We have found other isolated results. We feel, however, that those we present are the most interesting.

Throughout this section $k > 0$ and $m \geq 0$ denote integers. The restrictions on k and m given in the statement of each theorem ensure that the right side is an alpha expansion that does not include α^n and α^{n+1} for any n. This means that Theorem 2.4 applies. The following theorem gives finite expansions for $F_k\alpha^m$ and $L_k\alpha^m$.

Theorem 4.1. *Let $k > 0$ and $m \geq 0$ be integers. Then*

$$F_k\alpha^m = \begin{cases} \sum_{i=1}^{k/2} \alpha^{4i+m-k-2}, & k \text{ even, } m \geq k-2; \\ \sum_{i=1}^{(k-1)/2} \alpha^{4i+m-k-2} + \alpha^{m+k-1}, & k \geq 3 \text{ odd, } m \geq k-2. \end{cases}$$

$$L_k\alpha^m = \begin{cases} \alpha^{m-k} + \alpha^{m+k}, & k \text{ even, } m \geq k; \\ \sum_{i=1}^{k} \alpha^{2i+m-k-1}, & k \text{ odd, } m \geq k-1. \end{cases}$$

For our next theorem it is convenient to first define the following infinite expansion that we also require in the next section.

$$x(k)=\sum_{i=0}^{\infty}\alpha^{2ki}.$$

We can now give the (infinite) expansions for $\dfrac{\alpha^m}{F_k}$ and $\dfrac{\alpha^m}{L_k}$.

Theorem 4.2. *Let $k>0$ and $m\geq 0$ be integers. Then*

$$\frac{\alpha^m}{F_k}=\begin{cases}\left(\alpha^{k+m-1}+\alpha^{k+m+1}\right)x(k), & k \text{ even };\\ \Big(\alpha^{k+m-1}+\sum_{i=1}^{k-2}\alpha^{2i+k+m}+\alpha^{3k+m-1} & \\ \quad +\alpha^{3k+m+2}\Big)\,x(2k), & k\geq 3 \text{ odd }.\end{cases}$$

$$\frac{\alpha^m}{L_k}=\begin{cases}\left(\sum_{i=1}^{k}\alpha^{2i+k+m-1}\right)x(2k), & k \text{ even };\\ \alpha^{k+m}x(k), & k \text{ odd }.\end{cases}$$

5. Predictable Alpha Expansions II

In order to state the results of this section succinctly we first define twelve two-parameter sums. In these sums k and m represent positive integers without restriction. Of course when a summation symbol is employed we follow the usual convention that the sum in question is zero if the upper limit of summation is less than the lower limit. In order to conserve space we present only those results that can be expressed neatly in terms of these twelve sums.

The first group of four sums is

$$\begin{aligned}
e(k,m) &= \sum_{i=1}^{m}\alpha^{2i+k-m-1},\\
f(k,m) &= \alpha^{k-m}+\sum_{i=1}^{k-m-1}\alpha^{2i+k+m-1}+\sum_{i=1}^{m}\alpha^{2i+3k-m-2}+\alpha^{3k+m+1},\\
g(k,m) &= \alpha^{k-m}+\alpha^{k+m},\\
h(k,m) &= \sum_{i=1}^{m-1}\alpha^{2i+k-m-1}+\sum_{i=1}^{k-m}\alpha^{2i+k+m-2}+\alpha^{3k-m+1}+\alpha^{3k+m}.
\end{aligned}$$

The second group is

$$\begin{aligned}
p(k,m) &= \sum_{i=1}^{(m-2)/2} \alpha^{4i+k-m-2} + \sum_{i=1}^{k-m+1} \alpha^{2i+k+m-3} + \alpha^{3k-m+2} \\
&\quad + \sum_{i=1}^{(m-2)/2} \alpha^{4i+3k-m} + \alpha^{3k+m-1}, \\
q(k,m) &= \sum_{i=1}^{(m-1)/2} \alpha^{4i+k-m-2} + \alpha^{k+m-1}, \\
r(k,m) &= \sum_{i=1}^{(m+1)/2} \alpha^{4i+k-m-2} + \sum_{i=1}^{k-m} \alpha^{2i+k+m} + \sum_{i=1}^{(m-1)/2} \alpha^{4i+3k-m}, \\
s(k,m) &= \sum_{i=1}^{m/2} \alpha^{4i+k-m-2}.
\end{aligned}$$

The third group is

$$\begin{aligned}
t(k,m) &= \alpha^{k-m-1} + \alpha^{k-m+1} + \alpha^{k+m-1} + \alpha^{k+m+1}, \\
u(k,m) &= \alpha^{k-m-1} + \sum_{i=1}^{m-2} \alpha^{2i+k-m} + \alpha^{k+m-1} + \sum_{i=1}^{k-m-2} \alpha^{2i+k+m+1} \\
&\quad + \alpha^{3k-m+2} + \alpha^{3k+m-1} + \alpha^{3k+m+1}, \\
v(k,m) &= \alpha^{k-m-1} + \sum_{i=1}^{m-2} \alpha^{2i+k-m} + \alpha^{k+m-1} + \alpha^{k+m+2}, \\
w(k,m) &= \alpha^{k-m-1} + \alpha^{k-m+1} + \alpha^{k+m-1} + \sum_{i=1}^{k-m-2} \alpha^{2i+k+m} \\
&\quad + \alpha^{3k-m-1} + \sum_{i=1}^{m-2} \alpha^{2i+3k-m+1} + \alpha^{3k+m+2}.
\end{aligned}$$

Theorems 5.1–5.4 give the expansions for $\frac{F[m]}{F[k]}$, $\frac{L[m]}{L[k]}$, $\frac{F[m]}{L[k]}$, and $\frac{L[m]}{F[k]}$, respectively. Here, and for the remainder of this section, k and m are positive integers. The restrictions on k and m stem from two considerations. First, we require that any power of α is a non-negative integer, and that the upper limit in any summation is at least unity. Second, we require that each expansion be an alpha expansion that does not include α^n and α^{n+1} for any n. This means that Theorem 2.4 applies.

Theorem 5.1. *Let k and m be positive integers. Then*

$$\frac{F[m]}{F[k]} = \begin{cases} e(k,m)\,x(k), & m \text{ even, } k \geq m \text{ even;} \\ f(k,m)\,x(2k), & m \text{ odd, } k \geq m+2 \text{ odd;} \\ g(k,m)\,x(k), & m \text{ odd, } k \geq m+1 \text{ even;} \\ h(k,m)\,x(2k), & m \text{ even, } k \geq m+1 \text{ odd.} \end{cases}$$

Theorem 5.2. *Let k and m be positive integers. Then*

$$\frac{L[m]}{L[k]} = \begin{cases} f\,(k,m)\,x(2k), & m \text{ even, } k \geq m+2 \text{ even;} \\ e\,(k,m)\,x(k), & m \geq 3 \text{ odd, } k \geq m \text{ odd;} \\ h\,(k,m)\,x(2k), & m \geq 3 \text{ odd, } k \geq m+1 \text{ even;} \\ g\,(k,m)\,x(k), & m \text{ even, } k \geq m+1 \text{ odd.} \end{cases}$$

Theorem 5.3. *Let k and m be positive integers. Then*

$$\frac{F[m]}{L[k]} = \begin{cases} p\,(k,m)\,x(2k), & m \geq 4 \text{ even, } k \geq m \text{ even;} \\ q\,(k,m)\,x(k), & m \geq 3 \text{ odd, } k \geq m-2 \text{ odd;} \\ r\,(k,m)\,x(2k), & m \text{ odd, } k \geq m+1 \text{ even;} \\ s\,(k,m)\,x(k), & m \text{ even, } k \geq m-1 \text{ odd.} \end{cases}$$

Theorem 5.4. *Let k and m be positive integers. Then*

$$\frac{L[m]}{F[k]} = \begin{cases} t\,(k,m)\,x(k), & m \text{ even, } k \geq m+2 \text{ even;} \\ u\,(k,m)\,x(2k), & m \geq 3 \text{ odd, } k \geq m+4 \text{ odd;} \\ v\,(k,m)\,x(k), & m \geq 3 \text{ odd, } k \geq m+1 \text{ even;} \\ w\,(k,m)\,x(2k), & m \geq 4 \text{ even, } k \geq m+3 \text{ odd.} \end{cases}$$

Theorems 5.5–5.7 give the expansions for $\frac{F[m]\sqrt{5}}{F[k]}$, $\frac{L[m]\sqrt{5}}{L[k]}$, and $\frac{F[m]\sqrt{5}}{L[k]}$, respectively. We have found similar expansions for $\frac{L[m]\sqrt{5}}{F[k]}$, but since these cannot be expressed succinctly in terms of the twelve sums defined above, we have chosen not to present them here.

Theorem 5.5. *Let k and m be positive integers. Then*

$$\frac{F[m]\sqrt{5}}{F[k]} = \begin{cases} v\,(k,m)\,x(k), & m \geq 4 \text{ even, } k \geq m+2 \text{ even;} \\ w\,(k,m)\,x(2k), & m \geq 3 \text{ odd, } k \geq m+4 \text{ odd;} \\ t\,(k,m)\,x(k), & m \geq 3 \text{ odd, } k \geq m+1 \text{ even;} \\ u\,(k,m)\,x(2k), & m \geq 4 \text{ even, } k \geq m+3 \text{ odd.} \end{cases}$$

Theorem 5.6. *Let k and m be positive integers. Then*

$$\frac{L[m]\sqrt{5}}{L[k]} = \begin{cases} w\,(k,m)\,x(2k), & m \geq 4 \text{ even, } k \geq m+4 \text{ even;} \\ v\,(k,m)\,x(k), & m \geq 3 \text{ odd, } k \geq m+2 \text{ odd;} \\ u\,(k,m)\,x(2k), & m \geq 3 \text{ odd, } k \geq m+3 \text{ even;} \\ t\,(k,m)\,x(k), & m \text{ even, } k \geq m+1 \text{ odd.} \end{cases}$$

Theorem 5.7. *Let k and m be positive integers. Then*

$$\frac{F[m]\sqrt{5}}{L[k]} = \begin{cases} h\,(k,m)\,x(2k), & m \text{ even, } k \geq m+2 \text{ even;} \\ g\,(k,m)\,x(k), & m \text{ odd, } k \geq m \text{ odd;} \\ f\,(k,m)\,x(2k), & m \text{ odd, } k \geq m+3 \text{ even;} \\ e\,(k,m)\,x(k), & m \text{ even, } k \geq m-1 \text{ odd.} \end{cases}$$

Theorems 5.8 and 5.9 give the expansions for $\frac{F[m]}{F[k]\sqrt{5}}$ and $\frac{L[m]}{F[k]\sqrt{5}}$. For the same reason given above, we suppress the expansions for $\frac{L[m]}{L[k]\sqrt{5}}$. We have not been able to find alpha expansions of the type given in this section for $\frac{F[m]}{L[k]\sqrt{5}}$. The best that we have achieved for $\frac{F[m]}{L[k]\sqrt{5}}$ are one-parameter families (i.e. where only one of k or m is allowed to vary) that are not as striking as as those presented above. We have chosen, therefore, not to give them here.

Theorem 5.8. *Let k and m be positive integers. Then*

$$\frac{F[m]}{F[k]\sqrt{5}}=\begin{cases} s\,(k,m)\,x(k), & m\geq 4 \text{ even, } k\geq m-2 \text{ even;}\\ r\,(k,m)\,x(2k), & m \text{ odd, } k\geq m \text{ odd;}\\ q\,(k,m)\,x(k), & m\geq 3 \text{ odd, } k\geq m-1 \text{ even;}\\ p\,(k,m)\,x(2k), & m\geq 4 \text{ even, } k\geq m-1 \text{ odd.}\end{cases}$$

Theorem 5.9. *Let k and m be positive integers. Then*

$$\frac{L[m]}{F[k]\sqrt{5}}=\begin{cases} g\,(k,m)\,x(k), & m \text{ even, } k\geq m \text{ even;}\\ h\,(k,m)\,x(2k), & m\geq 3 \text{ odd, } k\geq m+2 \text{ odd;}\\ e\,(k,m)\,x(k), & m\geq 3 \text{ odd, } k\geq m-1 \text{ even;}\\ f\,(k,m)\,x(2k), & m \text{ even, } k\geq m+3 \text{ odd.}\end{cases}$$

6. A Sample Proof

The proofs of all results in Sections 4 and 5 follow similar lines. To illustrate, we prove Theorem 5.1 for the case where k and m are both odd.

For positive integers k and m, with $k \geq m+2$, we find closed forms for the values $f(k,m)$ and $x(2k)$ and obtain

$$\begin{aligned} f(k,m)x(2k) &= \frac{\alpha^{k-m}+\alpha^{k+m}+\alpha^{3k-m}\left(\alpha^{-1}-\alpha^{-2}\right)+\alpha^{3k+m}\left(\alpha-\alpha^{-1}\right)}{1-\alpha^{4k}}\\ &= \frac{\alpha^{k-m}+\alpha^{k+m}-\alpha^{3k-m}-\alpha^{3k+m}}{1-\alpha^{4k}}\\ &= \frac{\left(\alpha^{k-m}+\alpha^{k+m}\right)\left(1-\alpha^{2k}\right)}{\left(1+\alpha^{2k}\right)\left(1-\alpha^{2k}\right)}\\ &= \frac{\alpha^{k-m}+\alpha^{k+m}}{1+\alpha^{2k}}, \end{aligned}$$

which is true regardless of the parities of k and m.

Next, expressing β and γ in terms of α we find that the Binet form for F_n is

$$F_n=\frac{\alpha^{-n+1}+(-1)^{n+1}\alpha^{n+1}}{1+\alpha^2},$$

which, for k and m both odd, gives

$$\frac{F_m}{F_k}=\frac{\alpha^{-m+1}+\alpha^{m+1}}{\alpha^{-k+1}+\alpha^{k+1}}=\frac{\alpha^{k-m}+\alpha^{k+m}}{1+\alpha^{2k}}=f(k,m)x(2k).$$

7. Relationship with Work in Older Literature

Let $\{r_i\}_{i\geq 1}$ denote a non-increasing sequence of real numbers with limit zero, and let each of $\{k_i\}_{i\geq 1}$ and $\{m_i\}_{i\geq 1}$ denote a non-negative integer sequence. Write $S=\sum_{i\geq 1} k_i r_i$ and $S^*=\sum_{i\geq 1} m_i r_i$, where each of S or S^* may be finite or infinite. Then Fridy [2] defines $\{r_i\}_{i\geq 1}$ to be a $\{k,m\}$-base for the interval $(-S^*,S)$ if, for each $x\in(-S^*,S)$, there is an integer sequence $\{a_i\}_{i\geq 1}$ such that

$$x=\sum_{i\geq 1} a_i r_i, \text{ where } -m_i\leq a_i\leq k_i \text{ for each } i. \tag{7.1}$$

Fridy then proves two key theorems which, for the sake of completeness, we state next.

Theorem 7.1. *The sequence* $\{r_i\}_{i\geq 1}$ *is a* $\{k,m\}$*-base for* $(-S^*, S)$ *if and only if*

$$r_n \leq \sum_{i\geq n+1} (k_i + m_i)\, r_i \text{ for each } n. \tag{7.2}$$

Theorem 7.2. *The sequence* $\{r_i\}_{i\geq 1}$ *yields exactly one* $\{k,m\}$*-base representation for each* $x \in (-S^*, S)$ *if and only if*

$$r_n = \sum_{i\geq n+1} (k_i + m_i)\, r_i \text{ for each } n. \tag{7.3}$$

Fridy proved these two theorems after first proving the following key result, which he designated as a lemma (see pages 194–196).

Lemma 7.3. *The sequence* $\{r_i\}_{i\geq 1}$ *is a* $\{k,0\}$*-base for* $(0, S)$ *if and only if*

$$r_n \leq \sum_{i\geq n+1} k_i r_i \text{ for each } n. \tag{7.4}$$

In the later paper, Brown [1] establishes Fridy's key lemma by following a different path laid out much earlier by Kakeya. Furthermore, Brown extends Fridy's lemma by proving that an expansion

$$x = \sum_{i\geq 1} \beta_i r_i \tag{7.5}$$

for $x \in [0, S)$, with integers β_i satisfying $0 \leq \beta_i \leq k_i$ for $i \geq 1$, can be achieved with $\beta_i < k_i$ for infinitely many values of i. In addition, he proves that such an expansion is unique if and only if

$$r_n = \sum_{i\geq n+1} k_i r_i \text{ for } n \geq 1. \tag{7.6}$$

Brown concludes with several examples, including one example in which he proves that an arbitrary real number x with

$$-\sum_{i\geq 1} \frac{1}{F_i} \leq x \leq \sum_{i\geq 1} \frac{1}{F_i}$$

has an expansion of the form

$$x = \sum_{i\geq 1} \frac{\epsilon_i(x)}{F_i},$$

in which $\epsilon_i(x) \in \{-1, 1\}$.

To put the work that we present here in the context of the work of Fridy and Brown, consider Lemma 7.3 with $\{r_i\}_{i\geq 0} = \{\alpha^i\}_{i\geq 0}$, $\{k_i\}_{i\geq 0} = \{1\}_{i\geq 0}$, and $(0, S) = (0, 2+\alpha)$. Then our Theorem 2.2 follows as a consequence of Lemma 7.3 since

$$\alpha^n < \sum_{i\geq n+1} \alpha^i = \frac{\alpha^{n+1}}{1-\alpha} = \alpha^{n-1} \text{ for each } n \geq 0.$$

Our other theoretical results, however, are not subsumed by the work of Fridy or Brown. At this point we recall the well-known and widely available work of Edouard Zeckendorf [3]. Zeckendorf's main result states that each positive integer can be represented uniquely as a sum of non-adjacent Fibonacci numbers. Our work, of course, belongs to a setting that is different from that of Zeckendorf. However, to draw an analogy in the setting that we present there is a "non-adjacent" aspect in our Theorem 2.3, and a "uniqueness" aspect in our Theorem 2.4.

We would like to express our gratitude to an anonymous referee. This referee pointed out the work of Brown [1], and this led us to the work of Fridy [2]. As a result, the addition of Section 7 has served to better place our work in a historical context.

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