Abstract. We derive evaluations of the classical Dedekind sum $s(h,k)$ for certain classes of generalized Fibonacci and Lucas numbers that had previously not been considered. As particular cases we obtain explicit formulas for $s(p^n - 1, p^{n+1} - 1)$ for integers $p \geq 2$ and $n \geq 1$, and for $s(p^n + 1, p^{n+1} + 1)$, with $p \geq 2$ even.

1. Introduction

The classical Dedekind sum is defined by

$$s(d, c) = \sum_{j=1}^{c} \left( \left( \frac{j}{c} \right) \right) \left( \left( \frac{dj}{c} \right) \right),$$

with

$$\left( \left( x \right) \right) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - \lfloor x \rfloor - \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Because of important applications, mainly in number theory, the Dedekind sum has been studied extensively by many authors in a variety of contexts. See Rademacher and Grosswald [4] for a bibliography. The most important result about Dedekind sums, first proved by Dedekind himself [2], is the reciprocity law. There are many different proofs in the literature, including four in [4].

**Theorem 1** (Reciprocity Law). If $(h, k) = 1$ and $h, k > 0$, then

$$s(k, h) + s(h, k) = \frac{h^2 + k^2 + 1 - 3hk}{12hk}.$$  (1.1)

From the definition and with the help of this reciprocity law one can easily obtain special values of the Dedekind sum, among them

$$s(1, k) = \frac{(k - 1)(k - 2)}{12k},$$  (1.2)

and, if $k$ is odd,

$$s(2, k) = \frac{(k - 1)(k - 5)}{24k}.$$  (1.3)

The first author was supported in part by the Natural Sciences and Engineering Research Council of Canada.
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(see, e.g., [1, p. 62]), with numerous extensions that can be found in [1, p. 73] and more recently in [6]. A related property is the fact that the only integer value taken by $s(h, k)$ is zero, and

$$s(h, k) = 0 \quad \text{if and only if} \quad h^2 + 1 \equiv 0 \pmod{k}.$$  

The second author [3] used this and the reciprocity law (1.1) to show that $s(h, k) = s(k, h)$ if and only if $h = F_{2n+1}$ and $k = F_{2n+3}$ for positive integers $n$, where $F_n$ is the $n$th Fibonacci number.

This indicates that Fibonacci numbers play a special role in the evaluation of the Dedekind sum. Indeed, special values of the Dedekind sum can be found in [1, p. 72] and [7], with extensions to Lucas and generalized Lucas numbers in [5], and further extensions in [8].

It is the purpose of this paper to deal with several classes of generalized Fibonacci and Lucas sequences not covered by the results in [8]. In Section 2 we introduce the sequences to be considered, and in Section 3 we prove our evaluations. We conclude this paper with some easy consequences in Section 4.

2. The Sequences

The sequences under consideration here are as follows. For $p, q \in \mathbb{Z}$ we define the generalized Fibonacci and Lucas sequences, respectively, by

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = pu_n - qu_{n-1}$$

and

$$v_0 = 2, \quad v_1 = p, \quad v_{n+1} = pv_n - qv_{n-1}.$$  

If we set $\Delta = p^2 - 4q$, $\alpha = (p + \sqrt{\Delta})/2$, and $\beta = (p - \sqrt{\Delta})/2$, these sequences have the standard Binet forms, for $n \geq 0$,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n. \quad (2.1)$$

The classical Fibonacci and Lucas sequences are obtained when $p = 1$ and $q = -1$. In [5] and [8], the class of sequences investigated involve $|q| = 1$. In this paper we consider certain cases involving arbitrary values of $q$; in particular we address the sequences

$$u_{n+1} = pu_n + (p + 1)u_{n-1}, \quad v_{n+1} = pv_n + (p + 1)v_{n-1} \quad (2.2)$$

and

$$u_{n+1} = (p + 1)u_n - pu_{n-1}, \quad v_{n+1} = (p + 1)v_n - pv_{n-1}, \quad (2.3)$$

with initial values as above. Because the characteristic polynomials of these sequences are reducible, we get the following first-order recurrences.

**Lemma 1.** For the sequences given by (2.2) we have, respectively,

$$u_{n+1} = (p + 1)u_n + (-1)^n$$

and

$$v_{n+1} = (p + 1)v_n + (-1)^{n+1}(p + 2).$$

**Lemma 2.** For the sequences given by (2.3) we have, respectively,

$$u_{n+1} = pu_n + 1$$

and

$$v_{n+1} = pv_n - (p - 1).$$
The proofs of these lemmas are straightforward and follow directly from the Binet forms in (2.1).

3. The Evaluations

To establish our evaluations we require a few well-known elementary properties of the Dedekind sum, which we summarize in the following lemmas.

**Lemma 3.** For integers $h, k, h_1, h_2$, and $q$ we have the following.

(a) $s(-h, k) = -s(h, k)$ and $s(h, -k) = s(h, k)$.
(b) If $h_1 \equiv h_2 \pmod{k}$, then $s(h_1, k) = s(h_2, k)$.
(c) $s(qh, qk) = s(h, k)$.

The next lemma can be found in [1, p. 73] as Exercise 13.

**Lemma 4.** If $h, k, r \geq 1$, $(h, k) = 1$, and $k \equiv r \pmod{h}$, then

$$s(h, k) = \frac{h^2 + k^2 + 1 - (12s(r, h) + 3)hk}{12hk}.$$ 

We consider evaluations of the Dedekind sums at consecutive terms of the sequences under consideration. The proofs are similar to each other; hence we omit many of the calculations in later proofs.

**Theorem 2.** Let the sequence $\{u_n\}$ be defined by the first part of (2.2), with $p \in \mathbb{N}$. Then we have

$$s(u_{2n+1}, u_{2n+2}) = \frac{(p + 1)^2u_{2n}((p + 1)u_{2n} + p - 3) + p^2 - 3p + 2}{12u_{2n+2}}$$

and

$$s(u_{2n}, u_{2n+1}) = \frac{(p + 1)^2u_{2n-1}(p + 3 - (p + 1)u_{2n-1}) - p^2 - 3p - 2}{12u_{2n+1}}.$$ 

**Proof.** We prove the evaluation (3.1); the proof of (3.2) is similar. By the Reciprocity Law, we have

$$s(u_{2n+1}, u_{2n+2}) = \frac{u_{2n+1}^2 + u_{2n+2}^2 + 1 - 3u_{2n+1}u_{2n+2}}{12u_{2n+1}u_{2n+2}} - s(u_{2n+2}, u_{2n+1}).$$

Now, by the definition of $u_n$, Lemma 3(b), Lemma 1, Lemma 3(a), and Lemma 3(c), we have that

$$s(u_{2n+2}, u_{2n+1}) = s(pu_{2n+1} + (p + 1)u_{2n}, u_{2n+1})$$

$$= s((p + 1)u_{2n}, u_{2n+1})$$

$$= s(u_{2n+1} - 1, u_{2n+1})$$

$$= s(-1, u_{2n+1})$$

$$= -s(1, u_{2n+1})$$

$$= -\frac{(u_{2n+1} - 1)(u_{2n+1} - 2)}{12u_{2n+1}}.$$
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Then we get

\[ s(u_{2n+1}, u_{2n+2}) = \frac{u_{2n+1}^2 + u_{2n+2}^2 + 1 - 3u_{2n+1}u_{2n+2} + (u_{2n+1} - 1)(u_{2n+1} - 2)}{12u_{2n+1}u_{2n+2}} \]
\[ = \frac{u_{2n+1}^2 + u_{2n+2}^2 + 1 - 6u_{2n+1}u_{2n+2} + u_{2n+1}^2 u_{2n+2} + 2u_{2n+2}}{12u_{2n+1}u_{2n+2}} \]
\[ = \frac{(u_{2n+2} - pu_{2n+1})^2 + (1 - p^2)u_{2n+1}^2 + 1}{12u_{2n+1}u_{2n+2}} \]
\[ + \frac{(2p - 6)u_{2n+1}u_{2n+2} + u_{2n+1}^2 u_{2n+2} + 2u_{2n+2}}{12u_{2n+1}u_{2n+2}}. \]

where, prior to the last step, we subtract and add \(2pu_{2n+2}u_{2n+1} - p^2u_{2n+1}^2\) in the numerator. After some simplification, including several applications of Lemma 1, we arrive at the desired result.

**Theorem 3.** Let the sequence \(\{v_n\}\) be defined by the second part of (2.2), where \(p \in \mathbb{N}\) and \(p\) is odd. Then we have

\[ s(v_{2n+1}, v_{2n+2}) = -\frac{(p + 1)(v_{2n+1} - \frac{(p + 1)(p + 7)}{2} - \frac{p^2 + 9p + 12}{2})}{12(p + 2)v_{2n+2}} \]

and

\[ s(v_{2n}, v_{2n+1}) = \frac{(p + 1)v_{2n} \left( \frac{p^2 - 4p - 17}{2} \right) - (p + 3)(p^2 - 5p - 12)}{12(p + 2)v_{2n+1}}. \]

**Proof.** The proofs of these results follow the same steps as the proof of Theorem 2, but in this case we note that the initial calculation involves the evaluation of \(s(p + 2, v_{2n+1})\). This is accomplished with Lemma 4 and the evaluation of \(s(2, p + 2)\) from the identity (1.3).

The proofs of Theorems 4 and 5 are also analogous to that of Theorem 2 and are omitted.

**Theorem 4.** Let \(\{u_n\}\) be defined by the first part of (2.3), where \(p \in \mathbb{N}\). Then we have

\[ s(u_n, u_{n+1}) = -\frac{pu_n(u_n - p)}{12u_{n+1}}. \]

**Theorem 5.** Let \(\{v_n\}\) be defined by the second part of (2.3), where \(p \in \mathbb{N}\) and \(p\) is even. Then we have

\[ s(v_n, v_{n+1}) = \frac{p \left( v_n (p(p - 6) + 2v_n) - (p^2 - 7p + 4) \right)}{24(p - 1)v_{n+1}}. \]

4. SOME CONSEQUENCES

In this final brief section we show how Theorems 4 and 5, together with the Binet formula, easily provide some special evaluations involving powers of integers.

**Corollary 1.** For \(p \geq 2\) and \(n \in \mathbb{N}\), we have

\[ s \left( p^n - 1, p^{n+1} - 1 \right) = \frac{p(p^n - 1)(p^n - p^2 + p - 1)}{12(p^{n+1} - 1)(p - 1)}. \]
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Proof. The Binet form (2.1) for the sequence \( u_n \) is \( u_n = (p^n - 1)/(p - 1) \). In light of this and Lemma 3(c), we have that

\[
 s(u_n, u_{n+1}) = s \left( \frac{p^n - 1}{p - 1}, \frac{p^{n+1} - 1}{p - 1} \right) = s \left( p^n - 1, p^{n+1} - 1 \right). 
\]

We use the evaluation from Theorem 4 and the Binet formula again to get the desired result. \( \square \)

Corollary 2. For even \( p \geq 2 \) and \( n \in \mathbb{N} \), we have

\[
 s \left( p^n + 1, p^{n+1} + 1 \right) = \frac{p^n \left( 2p^n + p^2 - 6p + 4 \right) + p - 2}{24(p - 1)(p^{n+1} + 1)}. 
\]

Proof. This is a direct application of the Binet form \( v_n = p^n + 1 \) and Theorem 5. \( \square \)

References


MSC2010: 11F20, 11B39

DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, B3H 3J5, CANADA
E-mail address: dilcher@mathstat.dal.ca

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13144
E-mail address: jmeye01@syr.edu