SOME PROPERTIES OF CYCLIC COMPOSITIONS

ARNOLD KNOPFMACHER AND NEVILLE ROBBINS

ABSTRACT. Say that two compositions of n into k parts are related if they differ only by a cyclic shift. This defines an equivalence relation on the set of such compositions. Let $\left\langle {n \atop k} \right\rangle$ denote the number of distinct corresponding equivalence classes, that is, the number of cyclic compositions of n into k parts. We prove some theorems concerning $\left\langle {n \atop k} \right\rangle$.

1. INTRODUCTION

If k, n are integers such that $1 \leq k \leq n$, let $C(n,k)$ denote the set of all compositions of n into k parts. (It is well-known that the cardinality of $C(n,k)$ is $\binom{n-1}{k-1}$ [2]). Now consider the relation R defined on $C(n,k)$ as follows: two elements of $C(n,k)$ are related if and only if they differ only by a cyclic shift. For example, we have

$$[1+2+3]_R = [2+3+1]_R = [3+1+2]_R \neq [1+3+2]_R.$$

Let $\left\langle {n \atop k} \right\rangle$ denote the number of distinct equivalence classes of $C(n,k)$ with respect to R. We refer to $\left\langle {n \atop k} \right\rangle$ as the number of *cyclic compositions* of n with k parts. These combinatorial objects were first investigated more than a century ago. (See [5] and the references contained therein.)

Suppose that for each $n \geq 2$ and for each k such that $1 \leq k \leq n-1$, we list all the $\left\langle {n \atop k} \right\rangle$ in a horizontal row. We then obtain a triangular array similar to Pascal's triangle, namely

$$
\begin{array}{c}
1\\
1\ \ 1\\
1\ \ 2\ \ 1\\
1\ \ 2\ \ 2\ \ 1\\
1\ \ 3\ \ 4\ \ 3\ \ 1\\
1\ \ 3\ \ 5\ \ 5\ \ 3\ \ 1\\
1\ \ 4\ \ 7\ \ 10\ \ 7\ \ 4\ \ 1\\
1\ \ 4\ \ 10\ \ 14\ \ 14\ \ 10\ \ 4\ \ 1\\
1\ \ 5\ \ 12\ \ 22\ \ 26\ \ 22\ \ 12\ \ 5\ \ 1\\
1\ \ 5\ \ 15\ \ 30\ \ 42\ \ 42\ \ 30\ \ 15\ \ 5\ \ 1\\
1\ \ 6\ \ 19\ \ 43\ \ 66\ \ 80\ \ 66\ \ 43\ \ 19\ \ 6\ \ 1\\
1\ \ 6\ \ 22\ \ 55\ \ 99\ \ 132\ \ 132\ \ 99\ \ 55\ \ 22\ \ 6\ \ 1\\
1\ \ 7\ \ 26\ \ 73\ \ 143\ \ 217\ \ 246\ \ 217\ \ 143\ \ 73\ \ 26\ \ 7\ \ 1\\
\cdots
\end{array}
$$

Remark 1. *This table appears in* [4] *as* A037306*. Note that for aesthetic reasons we omit the additional column* $\left\langle {n \atop n} \right\rangle = 1$ *from each row above.*

In this note, we derive an explicit formula for $\left\langle {n \atop k} \right\rangle$. We also give two proofs that the triangular array shown above is symmetric, that is, $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k} \right\rangle$ for all k such that $1 \le k \le n-1$. (One proof is based on generating functions while the second is combinatorial.) We prove some additional parity-related results, including the following: $2 | \left\langle {2n \atop n} \right\rangle$ for all $n \ge 2$. (The latter result is analogous to a similar property of binomial coefficients stated in (3) below.)

2. Preliminaries

Theorem 1. *If* p *is prime and* $1 \le k \le p-1$*, then* $\left\langle {p \atop k} \right\rangle = \frac{1}{p}\binom{p}{k}$*. This theorem, proven by Sommerville in* [5]*, will be generalized in Corollary 1 below.*

Definition 1. *Let* $D(n)$ *denote the total number of cyclic equivalence classes of compositions of* n*, that is*

$$D(n) = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle .$$

It is known that

$$D(n) = -1 + \frac{1}{n} \sum_{d|n} \phi(d) 2^{\frac{n}{d}} .$$

(This follows from [2, p. 48].)

Definition 2. *Let $D^*(n)$ denote the total number of necklaces of n beads in two colors, say black and white. It is known that $D^*(n) = 1 + D(n)$ [4].*

Definition 3. *Let $D^{**}(n)$ denote the total number of necklaces of n beads in two colors, with the proviso that at least one bead must be black.*

Clearly, $D^{**}(n) = D(n)$, but we will prove more, namely, we will demonstrate a bijection between cyclic compositions of n into k parts and necklaces with beads of which k are black, the rest white, where $1 \le k \le n-1$.

Definition 4. *Let $t_2(n)$ denote the sum of the binary digits of the natural number n.*

Definition 5. *Let $o_2(n) = m$ if $2^m | n$ but $2^{m+1} \not| n$.*

Definition 6. *Let $[x]$ denote the integer part of the real number x.*

Identities. For all $n \ge 1$, we have:

$$\sum_{k\, odd} \binom{n}{k} = 2^{n-1} \tag{1}$$

$$o_2\left(\binom{n}{k}\right) = t_2(k) + t_2(n-k) - t_2(n) \tag{2}$$

$$2 \Big| \binom{2n}{n} \tag{3}$$

$$\binom{2^n - 1}{k} \equiv 1 \pmod 2 \quad for\ all\ k \tag{4}$$

$$\text{If } t \text{ is odd and } a > b \text{ then } o_2\left(\binom{2^a}{2^b t}\right) = o_2\left(\binom{2^{a-b}}{t}\right). \tag{5}$$

Remark 2. *Identity (2) follows from the formula*

$$o_p(n!) = \frac{n - t_p(n)}{p-1}$$

which may be found on p. 26 of [1]; (3), (4), and (5) follow from (2).

3. The Main Results

We begin with our main result, which is an explicit formula for $\left\langle {n \atop k} \right\rangle$.

Theorem 2. *If $1 \le k \le n$, then*

$$\left\langle {n \atop k} \right\rangle = \frac{1}{n} \sum_{j | \gcd(n,k)} \phi(j) \binom{n/j}{k/j}.$$

Proof. We use the general expression for the bivariate generating function of cycles of unlabeled combinatorial structures, as given in [2, p. 729], specialized to the case of cyclic compositions (the corresponding univariate generating function appears on [2, p. 48]). This gives

$$C(z,u) := \sum_{n\ge0}\sum_{k\ge0} \left\langle {n \atop k} \right\rangle u^k z^n = \sum_{j\ge1} \frac{\phi(j)}{j} \log\left(1 - \frac{u^j z^j}{1-z^j}\right)^{-1}.$$

Now

$$\begin{aligned} C(z,u) &= \sum_{j\ge 1} \frac{\phi(j)}{j}\left(\log(1-z^j) - \log(1-(1+u^j)z^j)\right) \\ &= \sum_{j\ge 1} \frac{\phi(j)}{j}\left(-\sum_{m\ge 1}\frac{z^{jm}}{m} + \sum_{m\ge 1}\frac{(1+u^j)z^{jm}}{m}\right) \\ &= \sum_{j\ge 1}\sum_{m\ge 1}\frac{\phi(j)}{jm} z^{jm}\left((1+u^j)^m - 1\right). \end{aligned}$$

Setting $n = jm$ we get

$$\left\langle {n \atop k} \right\rangle = \frac{1}{n}\sum_{j|n}\phi(j)[u^k]\left((1+u^j)^{n/j} - 1\right) = \frac{1}{n}\sum_{j|\gcd(n,k)}\phi(j)\binom{n/j}{k/j}.$$

□

As a corollary of Theorem 2 we obtain the following result, which generalizes Theorem 1. We also give a direct combinatorial proof thereof.

Corollary 1. *If* $1 \le k \le n$ *and* $(k,n) = 1$*, then*

$$\left\langle {n \atop k} \right\rangle = \frac{1}{n}\binom{n}{k}.$$

Proof. (*First.*) This follows directly from the hypothesis and Theorem 2. □

Proof. (*Second.*) Suppose that a cyclic equivalence class of compositions of n into k parts has order d. Then d must divide the order of the cyclic group of order k, that is, $d|k$. Then the composition, viewed as a sequence, has a subsequence of order $\frac{k}{d}$, and furthermore, $\frac{k}{d}|n$. Since $(k,n) = 1$ by hypothesis, this implies $\frac{k}{d} = 1$, that is, $d = k$. Since each of the $\left\langle {n \atop k} \right\rangle$ cyclic equivalence classes has order k, and since the total number of compositions of n into k parts is $\binom{n-1}{k-1}$, we have

$$k\left\langle {n \atop k} \right\rangle = \binom{n-1}{k-1} \rightarrow \left\langle {n \atop k} \right\rangle = \frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k}.$$

□

Another consequence of Theorem 2 is Corollary 2 below, which is analogous to the corresponding property of binomial coefficients given in (1).

Corollary 2.

$$\sum_{k\, odd}\left\langle {2^n \atop k} \right\rangle = 2^{2^n - n - 1}.$$

Proof. By the hypothesis and Theorem 2, we have

$$\left\langle {2^n \atop k} \right\rangle = \frac{1}{2^n}\binom{2^n}{k}$$

so that, invoking (1), we obtain:

$$\sum_{k\,odd} \left\langle {2^n \atop k} \right\rangle = \sum_{k\,odd} \frac{1}{2^n} \binom{2^n}{k} = \frac{1}{2^n} \sum_{k\,odd} \binom{2^n}{k} = \frac{1}{2^n}(2^{2^n-1}) = 2^{2^n-n-1}.$$

□

As a further consequence of Theorem 2, we obtain the symmetry property of cyclic composition coefficients, namely:

Theorem 3. *If* $1 \le k \le n-1$, *then* $\left\langle {n \atop n-k} \right\rangle = \left\langle {n \atop k} \right\rangle$.

Proof. (*First.*) This follows immediately from Theorem 2, to wit:

$$\left\langle {n \atop n-k} \right\rangle = \frac{1}{n} \sum_{j|(n-k,n)} \phi(j) \binom{n/j}{(n-k)/j} = \frac{1}{n} \sum_{j|(k,n)} \phi(j) \binom{n/j}{k/j} = \left\langle {n \atop k} \right\rangle.$$

□

We now offer an alternate proof of Theorem 3, based on combinatorics.

Proof. (*Second.*) There is a bijection between necklaces of n beads of which k are black, the rest white, and cyclic compositions of n into k parts as follows: Given such a necklace, start at any black bead. Let n_1 denote the number of beads traversed before coming to the second black bead. Let n_2 denote the number of beads traversed before coming to the third black bead, etc. Since there are a total of k black beads by hypothesis, we get a composition of n into k parts, namely $n = n_1 + n_2 + \cdots + n_k$. Conversely, given a composition of n into k parts, start with a black bead, add $n_1 - 1$ white beads, then a second black bead, then $n_2 - 1$ white beads, etc., finally $n_k - 1$ white beads to separate the kth and first black beads.

Note that each necklace with k black beads makes a contribution of 1 to $\left\langle {n \atop k} \right\rangle$ and each necklace with $n-k$ black beads makes a contribution of 1 to $\left\langle {n \atop n-k} \right\rangle$. But a necklace with $n-k$ black nodes also has k white nodes. Clearly, there is a bijection between necklaces with k black nodes and those with k white nodes. The conclusion now follows. □

Remark 3. *An alternate proof of Theorem 3 using lattice path representations of cyclic compositions was given by D. Wasserman [6].*

Let $C_n = \frac{1}{2n+1}\binom{2n}{n}$ *denote the nth Catalan number. Our triangular array of cyclic composition coefficients contains the Catalan numbers as the following theorem asserts.*

Theorem 4. $\left\langle {2n+1 \atop n} \right\rangle = C_n$ *for all* $n \ge 1$.

Proof. Since $(n, 2n+1) = 1$, Lemma 1 implies that

$$\left\langle {2n+1 \atop n} \right\rangle = \frac{1}{2n+1}\binom{2n+1}{n} = \left(\frac{1}{2n+1}\right)\frac{(2n+1)!}{n!(n+1)!}$$
$$= \left(\frac{1}{n+1}\right)\frac{(2n)!}{(n!)^2} = \left(\frac{1}{n+1}\right)\binom{2n}{n} = C_n.$$

□

There is also a connection between our triangular array of cyclic composition coefficients and the pentagonal numbers which play an important role in the theory of partitions. Recall that the pentagonal numbers are defined for $k \in Z$ by $\omega(k) = k(3k-1)/2$.

Theorem 5.

$$\left\langle {3n+1 \atop 3} \right\rangle = \omega(n); \quad \left\langle {3n+2 \atop 3} \right\rangle = \omega(-n).$$

Proof. Theorem 2 implies

$$\left\langle {3n+1 \atop 3} \right\rangle = \frac{1}{3n+1}\binom{3n+1}{3} = \frac{n(3n-1)}{2} = \omega(n).$$

Likewise

$$\left\langle {3n+2 \atop 3} \right\rangle = \frac{1}{3n+2}\binom{3n+2}{3} = \frac{n(3n+1)}{2} = \omega(-n).$$

□

4. Parity Considerations

The next theorem, which concerns the parity of the central cyclic composition coefficient, is analogous to a corresponding property of binomial coefficients given in (3) above:

Theorem 6. $2 \Big| \left\langle {2n \atop n} \right\rangle$ *for all* $n \geq 2$.

Proof. The second author has shown that $D(n) - 1$ is even for all $n > 2$ [3]. Thus, invoking the identity that follows Definition 1, and noting that $\left\langle {n \atop n} \right\rangle = 1$, we have

$$D(n) - 1 = \sum_{k=1}^{n-1} \left\langle {n \atop k} \right\rangle \quad \text{is even for all } n \geq 2.$$

Now

$$D(2n) - 1 = \sum_{k=1}^{2n-1} \left\langle {2n \atop k} \right\rangle = \sum_{k=1}^{n-1} \left\langle {2n \atop k} \right\rangle + \left\langle {2n \atop n} \right\rangle + \sum_{k=n+1}^{2n-1} \left\langle {2n \atop k} \right\rangle.$$

By Theorem 3 the two sums that appear in the right member of the last identity are equal. The conclusion now follows. □

The last theorem states a parity property of cyclic composition coefficients that again recalls a similar parity property of binomial coefficients.

Theorem 7.

$$(a) \left\langle {2^n - 1 \atop k} \right\rangle \equiv 1 \pmod 2 \text{ for } 1 \leq k \leq 2^n - 1.$$

$$(b) \left\langle {2^n + 1 \atop k} \right\rangle \equiv 0 \pmod 2 \text{ for } 2 \leq k \leq 2^n \text{ and } n \geq 2.$$

Proof. Theorem 2 implies

$$(2^n \pm 1) \left\langle {2^n \pm 1 \atop k} \right\rangle = \sum_{j|(k, 2^n \pm 1)} \phi(j) \binom{(2^n \pm 1)/j}{k/j}$$

$$\rightarrow \left\langle {2^n \pm 1 \atop k} \right\rangle \equiv \sum_{j|(k,2^n \pm 1)} \phi(j) \binom{(2^n \pm 1)/j}{k/j} \pmod 2.$$

Now $j|(2^n \pm 1) \Rightarrow j$ is odd $\Rightarrow \phi(j) \equiv 0 \pmod 2$ unless $j = 1$, so we have

$$\left\langle {2^n \pm 1 \atop k} \right\rangle \equiv \binom{2^n \pm 1}{k} \pmod 2.$$

Now (a) follows from (4). To prove (b), we must show that $\binom{2^n+1}{k} \equiv 0 \pmod 2$ for $2 \le k \le 2^n$, or equivalently, that $o_2\left(\binom{2^n+1}{k}\right) \ge 1$. Now (2) implies

$$o_2\left(\binom{2^n+1}{k}\right) = t_2(k) + t_2(2^n+1-k) - t_2(2^n+1) = t_2(k) + t_2(2^n+1-k) - 2.$$

If $t_2(k) \ge 2$, then $o_2\left(\binom{2^n+1}{k}\right) \ge t_2(2^n+1-k) \ge 1$. If $t_2(k) = 1$, then $k = 2^m$ and

$$o_2\left(\binom{2^n+1}{2^m}\right) = t_2(2^n+1-2^m) - 1.$$

If $t_2(2^n+1-2^m) = 1$, then $2^n+1-2^m = 2^r$, so $m = 0$ or $m = n$, and hence, $k = 1$ or $k = 2^n$, contrary to the hypothesis. Therefore $t_2(2^n+1-2^m) \ge 2$, so $o_2\left(\binom{2^n+1}{2^m}\right) \ge 1$. □

References

[1] R. D. Carmichael, *The Theory of Numbers*, John Wiley & Sons, 1914.

[2] P. Flajolet and R. Sedgwick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[3] N. Robbins, *Some congruence properties of the number of cyclic compositions*, (to appear in Japan J. of Algebra, Number Theory, and Applications).

[4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

[5] D. Y. M. Sommerville, *On certain perodic properties of cyclic compositions of integers*, Proc. London Math. Soc., (1909), 263–313.

[6] D. Wasserman, link from *A*037306 in *Online Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

MSC2010: 11B65, 11A07

School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa
E-mail address: `arnoldk@cam.wits.ac.za`

Department of Mathematics, San Francisco State University, San Francisco, CA 94132
E-mail address: `robbins@math.sfsu.edu`