ON RECURRENCES OF FAHR AND RINGEL: 
AN ALTERNATE APPROACH

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Abstract. In a recent article, Hirschhorn found the generating functions of two sequences introduced by Fahr and Ringel. We use a matrix method to obtain the same results in a simpler and more direct manner.

1. Introduction

In a recent paper, Fahr and Ringel [1] introduced two sequences $b_t[r]$ and $c_t[r]$ defined by the initial values $b_0[r] = c_0[r] = \delta_{r,0}$, and the recurrence relations

$$b_{t+1}[r] = c_t[r - 1] + 2c_t[r] - b_t[r],$$

$$c_{t+1}[t] = b_{t+1}[r] + 2b_{t+1}[r + 1] - c_t[r],$$

for $t, r \geq 0$, with the convention that $c_t[-1] = c_t[0]$. Their first few values are listed below.

<table>
<thead>
<tr>
<th>$b_t[r]$</th>
<th>$c_t[r]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>29</td>
</tr>
<tr>
<td>4</td>
<td>130</td>
</tr>
</tbody>
</table>

Define $B_r = B_r(q) = \sum_{t \geq 0} b_t[r]q^t$, and $C_r = C_r(q) = \sum_{t \geq 0} b_t[r]q^t$ as the “vertical” generating functions for $b_t[r]$ and $c_t[r]$. Hirschhorn [2] observed that

$$B_0 = \frac{1 + 3qC_0}{1 + q},$$

and, for $r \geq 0$,

$$B_{r+1} = \frac{1}{2} [(1 + q)C_r - B_r], \quad (1.1)$$

$$C_{r+1} = \frac{1}{2q} [(1 + q)B_{r+1} - qC_r]. \quad (1.2)$$

After rather lengthy computation that involves solving four auxiliary recurrences, he found explicit formulas for $B_r$ and $C_r$. The purpose of this short note is to use a simpler and more direct approach to derive the same results.
Equations (1.1) and (1.2) form a system of recurrences. Such a system can sometimes be solved rather effectively by a transfer matrix, as demonstrated by the author in [3]. For a detail discussion of the technique, see [6].

We first write (1.2) as

$$C_{r+1} = \frac{1+q}{2q} \left( \frac{1+q}{2} C_r - \frac{1}{2} B_r \right) - \frac{1}{2} C_r = -\frac{1+q}{4q} B_r + \frac{1+q^2}{4q} C_r,$$

(2.1)

so that (1.1) and (2.1) can be written in a matrix equation

$$\begin{bmatrix} B_{r+1} \\ C_{r+1} \end{bmatrix} = \frac{1}{4q} \begin{bmatrix} -2q & 2(1+q)q \\ -(1+q) & 1+q^2 \end{bmatrix} \begin{bmatrix} B_r \\ C_r \end{bmatrix}, \quad r \geq 0.
$$

Let $A$ denote the transfer matrix. It is clear that

$$\begin{bmatrix} B_r \\ C_r \end{bmatrix} = A^r \begin{bmatrix} B_0 \\ C_0 \end{bmatrix}$$

for $r \geq 0$. Hence,

$$\sum_{r \geq 0} \begin{bmatrix} B_r \\ C_r \end{bmatrix} x^r = \left( \sum_{r \geq 0} (xA)^r \right) \begin{bmatrix} B_0 \\ C_0 \end{bmatrix} = (I-xA)^{-1} \begin{bmatrix} B_0 \\ C_0 \end{bmatrix}.$$  

(2.2)

Finding the inverse of $I-xA$ is straightforward:

$$(I-xA)^{-1} = \frac{1}{4q} \left( \frac{1}{1-\frac{(1-q)^2}{4q} x + \frac{1}{4} x^2} \right) \begin{bmatrix} 4q - (1+q^2)x & 2(1+q)qx \\ -(1+q)x & 4q + 2qx \end{bmatrix}.$$  

Let

$$\mu = \frac{(1-q)^2 + (1+q)\sqrt{1-6q+q^2}}{8q} = \frac{1}{4q} (1-2q-3q^2-8q^3-\cdots),$$

$$\nu = \frac{(1-q)^2 - (1+q)\sqrt{1-6q+q^2}}{8q} = \frac{1}{4} (4q^2+28q^3+112q^4+\cdots),$$

so that

$$\frac{1}{1-\frac{(1-q)^2}{4q} x + \frac{1}{4} x^2} = \frac{1}{(1-\mu x)(1-\nu x)} = \sum_{r \geq 0} \frac{\mu^{r+1} - \nu^{r+1}}{\mu - \nu} x^r.$$  

After expanding the right-hand side of (2.2), and comparing the coefficients of $x^r$, we find

$$4q(\mu - \nu)B_r = 4qB_0(\mu^{r+1} - \nu^{r+1}) + [2(1+q)qC_0 - (1+q^2)B_0](\mu^r - \nu^r)$$

$$= [4qB_0\mu + 2(1+q)qC_0 - (1 + q^2)B_0] \mu^r$$

$$- [4qB_0\nu + 2(1+q)qC_0 - (1 + q^2)B_0] \nu^r,$$

$$4q(\mu - \nu)C_r = 4qC_0(\mu^{r+1} - \nu^{r+1}) + [2qC_0 - (1 + q)B_0](\mu^r - \nu^r)$$

$$= [4qC_0\mu + 2qC_0 - (1 + q)B_0] \mu^r - [4qC_0\nu + 2qC_0 - (1 + q)B_0] \nu^r.$$  

Recall that $B_r$ and $C_r$ are analytic infinite series in $q$, but $\mu$ has a pole at $q = 0$. Therefore, we need

$$4qB_0\mu + 2(1+q)qC_0 - (1+q^2)B_0 = 0, \quad (2.3)$$

$$4qC_0\mu + 2qC_0 - (1+q)B_0 = 0. \quad (2.4)$$

Both lead to

$$B_0 = \frac{1+q + \sqrt{1-6q+q^2}}{2} C_0.$$
Together with $B_0 = (1 + 3qC_0)/(1 + q)$, we find
\[ C_0 = \frac{(1 + q)\sqrt{1 - 6q + q^2} - (1 - 4q + q^2)}{2q(1 - 7q + q^2)}, \]
and
\[ B_0 = \frac{3\sqrt{1 - 6q + q^2} - (1 + q)}{2(1 - 7q + q^2)}, \]
as found by Hirschhorn [2]. They are also the generating functions for sequences A110122 and A132262, respectively, in OEIS [5]. Furthermore, from (2.3), we find
\[ -[4qB_0\mu + 2(1 + q)qC_0 - (1 + q^2)B_0] = -(4qB_0\nu - 4qB_0\mu) = 4q(\mu - \nu)B_0. \]
A similar result for $C_r$ can be derived from (2.4). From these we obtain the surprisingly simple main results of Hirschhorn:
\[ B_r = B_0\mu^r = B_0 \left( \frac{(1 - q)^2 - (1 + q)\sqrt{1 - 6q + q^2}}{8q} \right)^r, \]
\[ C_r = C_0\mu^r = C_0 \left( \frac{(1 - q)^2 - (1 + q)\sqrt{1 - 6q + q^2}}{8q} \right)^r. \]

3. Closing Remarks

There are other methods that one could use to find the generating functions. For instance, Prodinger [4] used bivariate generating functions and the kernel method to derive identical results.

References


MSC2010: 11B37, 11B39.

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