FIBONACCI NUMBERS, EULER'S 2-PERIODIC CONTINUED FRACTIONS AND MOMENT SEQUENCES

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Abstract. We prove that certain sequences of finite continued fractions associated with a 2-periodic continued fraction with period $a, b > 0$ are moment sequences of discrete signed measures supported in the interval $[-1, 1]$, and we give necessary and sufficient conditions in order that these measures are positive. For $a = b = 1$ this proves that the sequence of ratios F_{n+1}/F_{n+2}, $n \geq 0$, of consecutive Fibonacci numbers is a moment sequence.

1. Introduction

The motivation for this paper is the observation in Corollary 1.3: the sequence F_{n+1}/F_{n+2} of quotients of Fibonacci numbers is the moment sequence of a probability measure μ, i.e.,

$$F_{n+1}/F_{n+2} = \int x^n \, d\mu(x), \quad n = 0, 1, \ldots, \tag{1.1}$$

where μ is the discrete measure

$$\mu = \varphi\delta_1 + \sqrt{5}\sum_{k=1}^{\infty} \varphi^{4k} \delta_{(-\varphi^2)^k}. \tag{1.2}$$

Here $\varphi = (\sqrt{5} - 1)/2$ and δ_a denotes the Dirac measure having the mass 1 concentrated at the point $a \in \mathbb{R}$. We recall that $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Formula (1.1) can be proved easily from the so-called Binet formula

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n (1 - q^n), \tag{1.3}$$

where $q = (1 - \sqrt{5})/(1 + \sqrt{5})$. From this we get $(q = -\varphi^2)$

$$\begin{aligned} \frac{F_{n+1}}{F_{n+2}} &= \varphi \frac{1 - q^{n+1}}{1 - q^{n+2}} = \varphi \sum_{k=0}^{\infty} (1 - q^{n+1}) q^{(n+2)k} \\ &= \varphi + \varphi \sum_{k=1}^{\infty} (1 - q)(-q)^{2k-1} q^{nk} = \varphi + \sqrt{5} \sum_{k=1}^{\infty} \varphi^{4k} (-\varphi^2)^{nk}. \end{aligned}$$

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It is well-known that the difference equation $x_{n+1} = x_n + x_{n-1}$ of the Fibonacci numbers appears in the study of the continued fraction

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}\,. \tag{1.4}$$

In this paper we generalize the result above by studying the continued fraction

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\ddots}}}} \tag{1.5}$$

and the sequence $(s_n(a,b,w))$ of its *modified convergents* (to simplify the notation we remove the dependence on $a, b > 0$ and $w \geq 0$):

$$s_0 = w, \quad s_1 = \frac{1}{a+w}, \quad s_2 = \cfrac{1}{a+\cfrac{1}{b+w}}, \quad s_3 = \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+w}}}, \quad \cdots . \tag{1.6}$$

For $w = 0$ we obtain the sequence of convergents to (1.5).

Note that $s_n(a,a,1/a) = s_{n+1}(a,a,0)$ for $a > 0, n \geq 0$.

Our main result is Theorem 1.2 stating that the sequence $(s_n(a,b,w))$ is the moment sequence of a signed discrete measure ρ concentrated on the interval $[-1,1]$, i.e.,

$$s_n(a,b,w) = \int_{-1}^{1} x^n \, d\rho(x). \tag{1.7}$$

Since ρ is discrete, the integral in (1.7) is an infinite series. In addition we give a necessary and sufficient condition involving the parameters a, b, w in order that ρ is a positive measure. The result (1.1) appears as a special case.

Since this paper combines results about continued fractions with results about moment sequences, we give some explanation about these concepts.

The last chapter of Euler's masterpiece *Introductio in Analysin Infinitorum* (Vol. I), in English version [6], is devoted to continued fractions. Euler considered there 1- and 2-periodic continued fractions to get rational approximations to square roots of natural numbers. We say that a continued fraction of the form

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\ddots}}}} \tag{1.8}$$

is *k-periodic* if $a_{j+k} = a_j$ for $j = 1, 2, \ldots$.

If a natural number n is sum of two squares $n = m^2 + l^2$ of natural numbers, then for $a = 2m/l$ the convergents of the 1-periodic continued fraction

$$\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\ddots}}}} \tag{1.9}$$

give rational approximations to $\sqrt{n}$. Indeed, this continued fraction converges to a positive number x such that $x = 1/(a+x)$, i.e., it converges to the positive root of x^2+ax-1 which is

$$x = \frac{-a+\sqrt{a^2+4}}{2} = -\frac{m}{l} + \frac{\sqrt{n}}{l}.$$

For example, for $n = 5$, Euler took $m = 1$, $l = 2$, and so $a = 1$; the continued fraction (1.9) converges then to $(\sqrt{5}-1)/2$. In this example, the rational approximations are

$$\frac{0}{1},\quad \frac{1}{1},\quad \frac{1}{2},\quad \frac{2}{3},\quad \frac{3}{5},\quad \frac{5}{8},\quad \cdots \tag{1.10}$$

which are ratios of consecutive Fibonacci numbers F_n. For a relation between quotients of consecutive Fibonacci numbers and electrical networks see [14, p.43].

When n is not a sum of two squares, Euler considered a 2-periodic continued fraction of the form

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\ddots}}}} \tag{1.11}$$

to get rational approximations to $\sqrt{n}$. To do that, Euler was implicitly using *Pell's equation* $m^2 n = d^2 - 1$. Assuming that n is not a square of a natural number, Pell's equation has always infinitely many solutions $m, d \in \mathbb{N}$, cf. [9, p.210]. By taking positive integers a, b for which $ab = 2d-2$, the 2-periodic continued fraction above gives rational approximations to $\sqrt{n}$. Indeed, the continued fraction (1.11) converges to the positive root of $ax^2+abx-b$ given by

$$x = \frac{-ab+\sqrt{a^2b^2+4ab}}{2a} = -\frac{d-1}{a} + \frac{m\sqrt{n}}{a}.$$

For instance, for $n = 7$, Euler took $m = 3$, $d = 8$, $a = 2$, $b = 7$, and for these values of a, b the continued fraction (1.11) converges to $(-7+3\sqrt{7})/2$.

For the 2-periodic continued fraction (1.11) the sequence $s_n = s_n(a, b, w)$ defined in (1.6) satisfies

$$s_{n+2} = \cfrac{1}{a+\cfrac{1}{b+s_n}}, \tag{1.12}$$

and we leave to the reader to see by induction that s_n can be given by the formula

$$s_n = \frac{N_n}{D_n}, \quad n \geq 0, \tag{1.13}$$

where the sequences $(N_n)_n$ and $(D_n)_n$ are defined recursively by

$$N_{n+2} = bD_n + N_n, \quad n \geq 0, \tag{1.14}$$
$$D_{n+2} = abD_n + aN_n + D_n, \quad n \geq 0, \tag{1.15}$$

with the initial conditions $N_0 = w$, $N_1 = 1$, $D_0 = 1$, $D_1 = a + w$. By solving these difference equations we find an explicit formula for s_n leading to (1.7), see Section 2.

When $a = b$ the situation is simpler because (1.12) is replaced by

$$s_{n+1} = \frac{1}{a + s_n}, \tag{1.16}$$

and in this case

$$s_n = \frac{D_{n-1}}{D_n},$$

where D_n is the solution of the difference equation

$$D_{n+1} = aD_n + D_{n-1}, \quad n \geq 0, \tag{1.17}$$

with initial conditions $D_{-1} = w$, $D_0 = 1$.

The general difference equation of second order with constant coefficients $x_{n+1} = ax_n + bx_{n-1}$ has been studied by Kalman and Mena in [12].

Let us give a few comments about the special case (1.17). The parameter a in (1.17) can be parametrized $a = 2\sinh\theta, \theta > 0$ and in the form

$$x_{n+1} = 2\sinh\theta\, x_n + x_{n-1}, \quad n \geq 1, \tag{1.18}$$

which has been studied by Ismail in [11]. Using the initial conditions $x_0 = 0$, $x_1 = 1$, Ismail denoted the solution $F_n(\theta)$ and he called these numbers *generalized Fibonacci numbers*. They are natural numbers when $2\sinh\theta$ is a natural number, and the Fibonacci numbers correspond to $2\sinh\theta = 1$.

Let us now recall some facts about moment sequences. For details see [1]. The moments s_n, $n \geq 0$, of a positive (Borel) measure μ on the real line are defined as

$$s_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n \geq 0, \tag{1.19}$$

assuming that these integrals are finite.

Sequences of moments of positive measures were characterized by Hamburger in [8]. A necessary and sufficient condition is that all the Hankel matrices

$$\mathcal{H}_n = (s_{i+j})_{i,j=0}^n, \quad n = 0, 1, \ldots \tag{1.20}$$

are positive semidefinite.

Moment sequences of positive measures concentrated on $[0, \infty)$ were characterized by Stieltjes in his fundamental memoir [16]. In addition to (1.20) also the matrices

$$\mathcal{H}'_n = (s_{i+j+1})_{i,j=0}^n, \quad n = 0, 1, \ldots \tag{1.21}$$

shall be positive semidefinite.

It is remarkable that Stieltjes obtained his result long before Hamburger. For the early history of the moment problem see [13].

In [10] Hausdorff characterized moment sequences of positive measures concentrated on the unit interval $[0, 1]$ by complete monotonicity, i.e.,

$$(-1)^n(\Delta^n s)_k \geq 0 \quad \text{for all } n, k \geq 0,$$

where

$$(\Delta^0 s)_k = s_k,\ (\Delta s)_k = (\Delta^1 s)_k = s_{k+1} - s_k,\ (\Delta^n s)_k = (\Delta^{n-1}(\Delta s))_k,\ n \geq 2.$$

For a proof see also [1, 17, 4]. Moment sequences of positive measures on the interval $[-1, 1]$ can be characterized as bounded Hamburger moment sequences, i.e. bounded sequences (s_n) such that all the Hankel matrices (1.20) are positive semidefinite.

By the results above, moment sequences of positive measures form certain convex cones of real-valued sequences. Given a moment sequence of a positive measure one can ask if the measure is uniquely determined by the sequence. Stieltjes was the first to observe that this need not be the case, and we speak about an *indeterminate moment problem* in that case, see [1] for details. Here it is enough to mention that for a bounded moment sequence (1.19) the measure μ is uniquely determined and is concentrated on the interval $[-1, 1]$.

If we generalize (1.19) by considering signed measures on the interval $[-1, 1]$, then the corresponding set of moment sequences form a proper subspace M of the vector space $B(\mathbb{N}_0)$ of bounded real sequences. Since any signed measure is the difference of two positive measures, we see that each sequence in M is the difference of two bounded Hamburger moment sequences. It follows that each sequence from M is the moment sequence of a uniquely determined signed measure concentrated on the interval $[-1, 1]$.

We denote by $C([-1, 1])$ the Banach space of continuous functions $f : [-1, 1] \to \mathbb{R}$ with the supremum norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [-1, 1]\}.$$

The following result is an easy consequence of the fact that the dual space of $C([-1, 1])$ is the vector space of signed measures on $[-1, 1]$.

Proposition 1.1. *A sequence (c_n) of real numbers belongs to M if and only if there exists a constant $A > 0$ such that for all real polynomials $p(x) = \sum_{k=0}^n a_k x^k$*

$$\left|\sum_{k=0}^{n} a_k c_k\right| \leq A \|\sum_{k=0}^{n} a_k x^k\|_\infty.$$

We note in passing that the situation is different if one considers signed measures on the whole real line. By a result of Boas, cf. [17, p. 138], see also [5], any sequence of real numbers is a moment sequence of a signed measure, and it is even possible to choose a signed measure of the form $f(x)\,dx$ where f is a Schwartz function.

After these preliminaries we can now formulate our main theorem.

Theorem 1.2. *For $a, b, w \in \mathbb{R}$, $a, b > 0, w \geq 0$, the sequence $(s_n(a, b, w))$ defined in (1.6) is the sequence of moments of the discrete signed measure ρ supported in $[-1, 1]$ and defined by*

$$\rho = \frac{1}{a}(1-q)\delta_1 + \frac{1}{2a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty}\left(\alpha^{k+1} + \beta^{k+1}\right)\delta_{q^{k+1}} + \left(\alpha^{k+1} - \beta^{k+1}\right)\delta_{-q^{k+1}}, \tag{1.22}$$

where

$$q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2} \tag{1.23}$$

and

$$\alpha = \frac{q(aw - (1-q))}{qaw + 1 - q}, \quad \beta = \frac{qa - (1-q)w}{a + (1-q)w} \tag{1.24}$$

verify $0 < q < 1, |\alpha|, |\beta| < 1$.

Moreover, the measure ρ is positive if and only if the following conditions hold

(i) $a \geq b$,
(ii) $w \geq -b/2 + \sqrt{(b/2)^2 + b/a}$
(iii) $w \geq -(a+b)/4 + \sqrt{((a+b)/4)^2 + 1}$.

In particular for $a \geq b$ and $w = 1$, the measure ρ is always positive. Taking $a = b = 2\sinh\theta > 0$, and $w = 1/a$ we get the following result.

Corollary 1.3. *For any $\theta > 0$ the sequences*

$$\frac{F_{n+1}(\theta)}{F_{n+2}(\theta)}, \quad \text{and} \quad \frac{F_{n+3}(\theta)}{F_{n+2}(\theta)}, \quad n \geq 0 \tag{1.25}$$

of quotients of successive generalized Fibonacci numbers $F_n(\theta)$ defined by (1.18) are moment sequences of the positive discrete measures μ_θ and $\nu_\theta = (2\sinh\theta)\delta_1 + \mu_\theta$, where

$$\mu_\theta = e^{-\theta}\delta_1 + 2\cosh\theta \sum_{k=1}^{\infty} e^{-4k\theta}\delta_{(-e^{-2\theta})^k}. \tag{1.26}$$

In particular for $2\sinh\theta = 1$ the sequence $F_{n+1}/F_{n+2}, n \geq 0$, is the moments of the probability measure

$$\mu = \varphi\delta_1 + \sqrt{5}\sum_{k=1}^{\infty} \varphi^{4k}\delta_{(-\varphi^2)^k}, \tag{1.27}$$

where $\varphi = (\sqrt{5}-1)/2$.

Remark 1.4. Using Binet's formula (1.3) for the Fibonacci numbers, it is easy to see that $F_{n+1}, n \geq 0$, is a moment sequence of the measure

$$\tau = \frac{\sqrt{5}+1}{2\sqrt{5}}\delta_{(1+\sqrt{5})/2} + \frac{\sqrt{5}-1}{2\sqrt{5}}\delta_{(1-\sqrt{5})/2}.$$

A similar formula holds for the generalized Fibonacci numbers of Ismail, see [11, formula (2.2)].

Remark 1.5. It was proved in [3] that $F_\alpha/F_{n+\alpha}$, $n \geq 0$ is the moment sequence of a signed measure μ_α on $[-1, 1]$ and with total mass 1. Here α is a natural number and the signed measure μ_α is a probability measure precisely when α is an even number. The orthogonal polynomials corresponding to μ_α are little q-Jacobi polynomials, where $q = (1-\sqrt{5})/(1+\sqrt{5})$. Little q-Jacobi polynomials belong to the q-Askey scheme of orthogonal polynomials, and they are treated in [7]. The results above were used to prove Richardson's formula for the elements in the inverse of the Filbert matrix $(1/F_{1+i+j})$, cf. [15]. These results were extended to generalized Fibonacci numbers in [11]. For an extension to quantum integers see [2].

2. Proofs

Proof of Theorem 1.2. First of all, we find a closed expression for the denominators D_n of the modified convergents s_n defined in (1.6).

From (1.14) and (1.15) we get

$$D_{n+2} = a(bD_n + N_n) + D_n = aN_{n+2} + D_n,$$

hence

$$N_n = \frac{D_n - D_{n-2}}{a}, \quad n \geq 2, \tag{2.1}$$

so if D_n is determined, then we can find a formula for N_n and finally for s_n because of (1.13). The formula (2.1) can be extended to $n = 0, 1$ by defining $D_{-2} = 1 - aw$ and $D_{-1} = w$. Inserting (2.1) in (1.15), we find that

$$D_{n+2} = (2 + ab)D_n - D_{n-2}, \quad n \geq 0, \tag{2.2}$$

with initial conditions $D_{-2} = 1 - aw$, $D_{-1} = w$, $D_0 = 1$ and $D_1 = a + w$.

That means that the sequences (D_{2n}) and (D_{2n+1}) are both solutions of the difference equation

$$x_{n+1} = (2 + ab)x_n - x_{n-1}, \quad n \geq 0,$$

with initial conditions $x_{-1} = 1 - aw, w$, $x_0 = 1, a + w$, respectively. Any solution of this difference equation has the form $c_0 q_0^n + c_1 q_1^n$, where q_0 and q_1 are the solutions of $x^2 - (2 + ab)x + 1 = 0$. We write

$$q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2}, \tag{2.3}$$

so that q and $1/q$ are the solutions of $x^2 - (2 + ab)x + 1 = 0$, and $0 < q < 1$. We then know that there exist numbers c_0, c_1, d_0, d_1 such that

$$D_{2n} = c_0 q^{-n} + c_1 q^n, \quad n \geq -1, \tag{2.4}$$

$$D_{2n+1} = d_0 q^{-n} + d_1 q^n, \quad n \geq -1. \tag{2.5}$$

Using the initial conditions $D_{-2} = 1 - aw$, $D_{-1} = w$, $D_0 = 1$ and $D_1 = a + w$, we get two systems of linear equations with two unknowns, and solving them we find

$$c_0 = \frac{1 - q + qaw}{1 - q^2}, \qquad c_1 = \frac{q(1 - q - aw)}{1 - q^2}, \tag{2.6}$$

$$d_0 = \frac{a + (1 - q)w}{1 - q^2}, \qquad d_1 = \frac{q((1 - q)w - qa)}{1 - q^2}. \tag{2.7}$$

Note that $c_0, d_0 > 0$. Writing $\alpha = -c_1/c_0$ and $\beta = -d_1/(qd_0)$, we find

$$\alpha = \frac{q(aw - (1 - q))}{qaw + 1 - q}, \quad \beta = \frac{qa - (1 - q)w}{a + (1 - q)w}, \tag{2.8}$$

and it is clear that $|\alpha|, |\beta| < 1$ because $a > 0$, $0 < q < 1$ and $w \geq 0$.

We need to establish some technical properties of α and β, which we collect in the following lemma.

Lemma 2.1.

(1) *$\alpha \geq 0$ if and only if $w \geq -b/2 + \sqrt{(b/2)^2 + b/a}$.*
(2) *Assume $w > 0$. Then $-\beta \leq \alpha$ if and only if $a \geq b$.*
(3) *If $w = 0$ then $\beta = -\alpha = q$.*
(4) *$\alpha \geq \beta$ if and only if $w \geq -(a + b)/4 + \sqrt{((a + b)/4)^2 + 1}$.*

Proof. 1. By (2.8) we have that $\alpha \geq 0$ is equivalent to $q + aw - 1 \geq 0$, hence to

$$1 - aw \leq q = \frac{2 + ab - \sqrt{a^2b^2 + 4ab}}{2},$$

or

$$\sqrt{a^2b^2 + 4ab} \leq 2aw + ab,$$

which is equivalent to $aw^2 + abw - b \geq 0$ because $a > 0$. Since $a, b > 0$ and $w \geq 0$, we finally get that $\alpha \geq 0$ if and only if

$$w \geq -b/2 + \sqrt{(b/2)^2 + b/a}.$$

2. According to (2.8), $-\beta \leq \alpha$ if and only if

$$\frac{(1-q)w - aq}{a + (1-q)w} \leq \frac{q(aw - (1-q))}{qaw + 1 - q},$$

and a straightforward computation gives that this is equivalent to

$$w(1+q)(q^2 - (2+a^2)q + 1) \leq 0.$$

Using that q satisfies $q^2 - (2+ab)q + 1 = 0$, the left-hand side of this inequality can be reduced to $w(1+q)aq(b-a)$, and for $w > 0$ the result follows.

3. Follows by inspection.
4. We similarly get that $\alpha \geq \beta$ if and only if

$$w^2 + w(a+b)/2 - 1 \geq 0,$$

which is equivalent to the given condition because $w \geq 0$. □

We now continue the proof of Theorem 1.2.

Using (2.1), we can write

$$s_n = \frac{N_n}{D_n} = \frac{1}{a}\frac{D_n - D_{n-2}}{D_n} = \frac{1}{a}\left(1 - \frac{D_{n-2}}{D_n}\right).$$

From the formulas (2.4) and (2.6), and taking into account that $\alpha = -c_1/c_0$, we have

$$\begin{aligned}
\frac{D_{2n-2}}{D_{2n}} &= \frac{c_0 q^{-n+1} + c_1 q^{n-1}}{c_0 q^{-n} + c_1 q^n} = q\frac{1 - \alpha q^{2n-2}}{1 - \alpha q^{2n}} \\
&= q(1 - \alpha q^{2n-2}) \sum_{k=0}^{\infty} \alpha^k q^{2nk} \\
&= q\left(1 + (1 - q^{-2}) \sum_{k=0}^{\infty} \alpha^{k+1} q^{2n(k+1)}\right),
\end{aligned}$$

showing that $s_{2n} = \int t^{2n} d\mu$, $n \geq 0$, where the measure μ is defined as

$$\mu = \frac{1}{a}(1-q)\delta_1 + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \alpha^{k+1}\delta_{q^{k+1}}. \tag{2.9}$$

In a similar way, it can be proved that $s_{2n+1} = \int t^{2n+1} d\nu$, $n \geq 0$, where the measure ν is defined as

$$\nu = \frac{1}{a}(1-q)\delta_1 + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \beta^{k+1}\delta_{q^{k+1}}. \tag{2.10}$$

Take now the reflected measures $\check{\mu}$ and $\check{\nu}$ of μ and ν with respect to the origin:

$$\check{\mu} = \frac{1}{a}(1-q)\delta_{-1} + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \alpha^{k+1}\delta_{-q^{k+1}},$$

$$\check{\nu} = \frac{1}{a}(1-q)\delta_{-1} + \frac{1}{a}\left(\frac{1}{q} - q\right)\sum_{k=0}^{\infty} \beta^{k+1}\delta_{-q^{k+1}}.$$

The measure $\rho = (\mu + \check{\mu})/2 + (\nu - \check{\nu})/2$ then has the same even moments as μ and the same odd moments as ν. That is, the nth moment of ρ is just s_n, $n \geq 0$. A simple computation shows that the measure ρ is given by (1.22).

It is clear that the measure ρ is positive if and only if

$$\alpha^{k+1} + \beta^{k+1},\ \alpha^{k+1} - \beta^{k+1} \geq 0, \quad k \geq 0,$$

i.e., if and only if $\alpha \geq 0$ and $-\alpha \leq \beta \leq \alpha$. The second part of Theorem 1.2 follows now by applying Lemma 2.1. $\square$

Proof of Corollary 1.3. For $a = b = 2\sinh\theta > 0$ and $w = 1/a$ we see that the three conditions of Theorem 1.2 are satisfied, so $s_n = s_n(a, a, 1/a)$ is a moment sequence of a positive measure $\rho = \mu_\theta$. We prove that $s_n = F_{n+1}(\theta)/F_{n+2}(\theta)$ by induction. This formula holds for $n = 0$ by inspection and clearly $s_{n+1} = 1/(a + s_n)$. We therefore find, assuming the formula for a fixed n

$$s_{n+1} = \frac{1}{a + F_{n+1}(\theta)/F_{n+2}(\theta)} = \frac{F_{n+2}(\theta)}{aF_{n+2}(\theta) + F_{n+1}(\theta)} = \frac{F_{n+2}(\theta)}{F_{n+3}(\theta)}.$$

A small calculation shows that $\alpha = -\beta = q^2$, $q = e^{-2\theta}$, $e^{-\theta} = (1-q)/a$ and $\mu_\theta = \rho$ defined in (1.22) is given by (1.26). By the difference equation (1.18) we find

$$\frac{F_{n+3}(\theta)}{F_{n+2}(\theta)} = 2\sinh\theta + \frac{F_{n+1}(\theta)}{F_{n+2}(\theta)},$$

so also $F_{n+3}(\theta)/F_{n+2}(\theta)$ is a moment sequence and the corresponding measure ν_θ is given as $2\sinh\theta\,\delta_1 + \mu_\theta$. $\square$

3. Concluding remarks

Euler did not use 3-periodic continued fractions (nor any other periodicity bigger than 2) to find rational approximations of square roots of natural numbers: to use 3-periodic continued fractions is not more useful than to use 1-periodic continued ones, and to use 4-periodic continued fractions is not more useful than to use 2-periodic ones, and so forth, cf. [6, p.322].

One can consider modified convergents s_n like (1.6) for a 3-periodic continued fraction of positive periods a, b, c. Using the same approach as before, we can find three signed measures μ_0, μ_1 and μ_2 on $[-1, 1]$ such that the $(3n+i)$th moment of μ_i is equal to s_{3n+i}, $i = 0, 1, 2$, $n \geq 0$. However, we are not able to construct a measure on $\mathbb{R}$ from these three measures having its $(3n+i)$th moment equal to the nth moment of μ_i, $i = 0, 1, 2$. The same happens for k-periodicity when $k > 2$.

By Hamburger's theorem it follows that $\det(\mathcal{H}_n) \geq 0$ for all the Hankel matrices of moments of a positive measure, see (1.20).

Computations indicate that if we consider the sequence $(s_n(a, a, c, 1))$, then some of the above determinants are negative except when $c = a$, so it does not seem possible to extend our results to k-periodic continued fractions for $k > 2$.

References

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, English translation, Oliver and Boyd, Edinburgh, 1965.

[2] J. E. Andersen and C. Berg, *Quantum Hilbert matrices and orthogonal polynomials*, J. Comput. Appl. Math., **233** (2009), 723–729.

[3] C. Berg, *Fibonacci numbers and orthogonal polynomials*, ArXiv:math.NT/0609283.

[4] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic analysis on semigroups. Theory of positive definite and related functions*, Graduate Texts in Mathematics, Vol. **100**, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[5] A. J. Durán, *The Stieltjes moments problem for rapidly decreasing functions*, Proc. Amer. Math. Soc., **107** (1989), 731–741.

[6] L. Euler, *Introduction to Analysis of the Infinite*, Book I, Springer-Verlag, New York, 1988.
[7] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge 1990, second edition 2004.
[8] H. Hamburger, *Über eine erweiterung des Stieltjesschen Momentenproblems*, Math. Ann., **81** (1920), 235–319, **82** (1921), 120–164, 168–187.
[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Oxford University Press, 1960.
[10] F. Hausdorff, *Momentenprobleme für ein endliches Intervall*, Math. Z., **16** (1923), 220–248.
[11] M. E. H. Ismail, *One Parameter Generalizations of the Fibonacci and Lucas Numbers*, The Fibonacci Quarterly, **46/47.2** (2008/2009), 167–180.
[12] D. Kalman and R. Mena, *The Fibonacci numbers-exposed*, Math. Mag., **76** (2003), 167–181.
[13] T. H. Kjeldsen, *The early history of the moment problem*, Historia Math., **20** (1993), 19–44.
[14] T. Koshy, *Fibonacci and Lucas Numbers With Applications*, John Wiley, New York, 2001.
[15] T. M. Richardson, *The Filbert matrix*, The Fibonacci Quarterly, **39.3** (2001), 268–275.
[16] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, **8** (1894), 1–122; **9** (1895), 5–47.
[17] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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