Abstract. In this paper we obtain a closed-form expression for the sum of the elements lying on the \( n \)th diagonal of a Fibonacci triangle. This is achieved by obtaining and then utilizing the ordinary generating functions of two subsequences of the sequence of diagonal sums.

1. Introduction

It is well-known that the \( n \)th ‘diagonal sum’ of Pascal’s triangle is equal to \( F_n \); see [4] and [6]. Note that the \( n \)th diagonal comprises \( \left\lfloor \frac{n+1}{2} \right\rfloor \) elements, where \( \lfloor x \rfloor \) is the floor function, denoting the largest integer not exceeding \( x \).

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
& & & & & & & & \\
\end{array}
\]

For example, highlighted in Pascal’s triangle above are the 5th and 8th diagonals. We have

\[ 1 + 3 + 1 = 5 = F_5 \quad \text{and} \quad 1 + 6 + 10 + 4 = 21 = F_8. \]

In this article we consider the corresponding situation for the Fibonacci triangle \( \mathcal{T} \) shown below, which was discussed recently in [3] in connection with an infinite matrix of 0’s and 1’s that had been constructed from the Zeckendorf representations of the non-negative integers. This triangle also appears as A058071 in Sloane’s On-line Encyclopedia of Integer Sequences [8]. Furthermore, some of its properties were studied in [5] and [7].

The entry in position \( p \) (taken from left to right) of the \( r \)th row of \( \mathcal{T} \) is equal to \( F_pF_{r-p+1} \). From [2], [3] and [6] we know that the \( r \)th row-sum of \( \mathcal{T} \) is given by

\[
\sum_{p=1}^{r} F_pF_{r-p+1} = \frac{1}{5} (rF_{r+1} + 2(r + 1)F_r).
\]

This is the sequence of Fibonacci numbers convolved with themselves, and appears in [8] as A001629.
We are interested here in studying the $n$th diagonal sum $A_n$ of $T$. The two examples highlighted in $T$ above show that $A_6 = 8 + 3 + 2 = 13$ and $A_9 = 34 + 13 + 10 + 6 + 5 = 68$. It is clear that $A_n$ is equal either to

$$F_1 F_n + F_2 F_{n-2} + F_3 F_{n-4} + \ldots + F_{\left\lfloor \frac{n+1}{2} \right\rfloor} F_{n-\left\lfloor \frac{n+1}{2} \right\rfloor},$$

or to

$$F_1 F_n + F_2 F_{n-2} + F_3 F_{n-4} + \ldots + F_{\frac{n}{2}} F_{\frac{n}{2}},$$

depending on whether $n$ is odd or even, respectively. In fact, more generally it is possible to write

$$A_n = F_1 F_n + F_2 F_{n-2} + F_3 F_{n-4} + \ldots + F_{\left\lfloor \frac{n+1}{2} \right\rfloor} F_{n-\left\lfloor \frac{n+1}{2} \right\rfloor}$$

$$= \sum_{p=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} F_p F_{n-2(p-1)}. \quad (1.1)$$

Let us term the above expression a ‘semi-stretched convolution’. In this paper we obtain various ordinary generating functions associated with the sequence $\{A_n\}$ and hence a closed-form expression for $A_n$.

2. Generating Functions

It is easily checked that the first few terms of $\{A_n\}$ are given by:

$$1, 1, 3, 4, 9, 13, 25, 38, 68, 106, \ldots$$

Our aim in this section is to obtain the ordinary generating function $R(x)$ for $\{A_n\}$,

$$R(x) = A_1 x + A_2 x^2 + A_3 x^3 + \cdots,$$

and two further generating functions associated with $\{A_n\}$. 

52 VOLUME 49, NUMBER 1
It is actually very straightforward to calculate $R(x)$. Let $G(x)$ be the ordinary generating function for the Fibonacci numbers. Then from [6] we know that

$$G(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots$$

$$= \frac{x}{1 - x - x^2}$$

$$= \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right),$$

where

$$\phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2}.$$ 

Now, noting that

$$R(x) = (F_1 x + F_2 x^3 + F_3 x^5 + \cdots) (F_1 + F_2 x + F_3 x^2 + \cdots),$$

we have

$$R(x) = \frac{1}{x} G(x^2) \cdot \frac{1}{x} G(x)$$

$$= \frac{1}{5x^2} \left( \frac{1}{1 - \phi x^2} - \frac{1}{1 - \hat{\phi} x^2} \right) \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right).$$

However, it turns out that this is not particularly amenable with respect to finding a closed-form expression for $A_n$, and the alternative approach we adopt here is to consider $\{A_n\}$ as two interleaved sequences, $\{B_n\}$ and $\{C_n\}$, such that $B_n = A_{2n-1}$ and $C_n = A_{2n}$ for $n \geq 1$. Let us now obtain the ordinary generating functions for $\{B_n\}$ and $\{C_n\}$.

First,

$$F_0 + F_2 x^2 + F_4 x^4 + \cdots = \frac{1}{2} (G(x) + G(-x))$$

$$= \frac{1}{2\sqrt{5}} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} + \frac{1}{1 + \phi x} - \frac{1}{1 + \hat{\phi} x} \right)$$

$$= \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi^2 x^2} - \frac{1}{1 - \hat{\phi}^2 x^2} \right).$$

Thus it is the case that the generating function $Q_{\text{even}}(x)$ for the even-numbered Fibonacci numbers is given by

$$F_0 + F_2 x + F_4 x^2 + \cdots = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right).$$

From this it follows, on using the semi-stretched convolution (1.1), that the generating function

$$V(x) = C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

for $\{C_n\}$ may be expressed as

$$V(x) = G(x) Q_{\text{even}}(x)$$

$$= \frac{1}{5x} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right) \left( \frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right).$$

Similarly, since

$$F_1 x + F_3 x^3 + F_5 x^5 + \cdots = \frac{1}{2} (G(x) - G(-x)),$$
it may be shown that the generating function $Q_{odd}(x)$ for the odd-numbered Fibonacci numbers is given by

$$F_1 + F_3 x + F_5 x^2 + \cdots = \frac{1}{\sqrt{5}} \left( \frac{\phi}{1 - \phi^2 x} - \frac{\hat{\phi}}{1 - \hat{\phi}^2 x} \right),$$

and hence that the generating function

$$U(x) = B_1 x + B_2 x^2 + B_3 x^3 + \cdots$$

for $\{B_n\}$ is

$$U(x) = \frac{1}{5} \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right) \left( \frac{\phi}{1 - \phi^2 x} - \frac{\hat{\phi}}{1 - \hat{\phi}^2 x} \right). \quad (2.2)$$

Both $U(x)$ and $V(x)$ will be utilized in Section 3. Incidentally, we may retrieve $R(x)$ from these generating functions as follows:

$$R(x) = \frac{1}{x} U(x^2) + V(x^2)$$

$$= \frac{1}{5x^2} \left( \frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right) \left( \frac{1 + \phi x}{1 - \phi^2 x^2} - \frac{1 + \hat{\phi} x}{1 - \hat{\phi}^2 x^2} \right)$$

$$= \frac{1}{5x^2} \left( \frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right) \left( \frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right).$$

3. A Formula For $A_n$

Theorem 3.1.

$$A_n = \frac{1}{2} \left( F_{n+3} - F_2 \left[ \frac{n}{2} \right] - \left[ \frac{n-1}{2} \right] \right).$$

Proof. We start by obtaining a formula for $C_n$. The right-hand side of (2.1) is multiplied out and then, employing the method of partial fractions, is expressed in the form

$$\frac{1}{5x} \left( \frac{a}{1 - \phi^2 x} + \frac{b}{1 - \hat{\phi}^2 x} + \frac{c}{1 - \phi x} + \frac{d}{1 - \hat{\phi} x} \right)$$

for some $a, b, c, d \in \mathbb{R}$. Subsequently, by expanding each term as a power series in $x$, comparing coefficients on both sides of (2.1) and using the results

$$F_n = \frac{1}{\sqrt{5}} (\phi^n - \hat{\phi}^n) \quad \text{and} \quad F_n + 2F_{n-1} = \phi^n + \hat{\phi}^n,$$

which may be found in [1] and [6], it can be shown that

$$C_n = \frac{1}{2} (F_{2n+3} - F_{n+3}).$$

Adopting a similar method with (2.2) leads to the result

$$B_n = \frac{1}{2} (F_{2n+2} - F_{n+1}).$$

From these expressions for $B_n$ and $C_n$ it does indeed follow that

$$A_n = \frac{1}{2} \left( F_{n+3} - F_2 \left[ \frac{n}{2} \right] - \left[ \frac{n-1}{2} \right] \right).$$
FIBONACCI DIAGONALS

To take an example,

\[ A_8 = \frac{1}{2} \left( F_{11} - F_2 \left\lfloor \frac{8}{2} \right\rfloor - \left\lfloor \frac{4}{2} \right\rfloor \right) \]
\[ = \frac{1}{2} (F_{11} - F_7) \]
\[ = 38. \]

We note here that the sequence \( \{A_n\} \) does not appear in [8].

4. FURTHER COMMENTS

First, as is noted in [7], the following recurrence relations, valid for \( n \geq 1 \), follow very easily from the structure of \( T \):

\[ A_{2n+1} = A_{2n} + A_{2n-1} + F_{n+1} \quad \text{and} \quad A_{2n+2} = A_{2n+1} + A_{2n}, \]

where \( A_1 = A_2 = 1 \).

Next, it is interesting that both \( \{B_n\} \) and \( \{C_n\} \) have mathematical lives of their own. We state here, without proof, a number of results associated with these sequences. The interested reader might like to consult [8] in this regard, where \( \{B_n\} \) and \( \{C_n\} \) appear as A094292 and A056014, respectively.

The sequence \( \{B_n\} \) is associated with a particular one-dimensional random walk. Indeed, \( B_n \) gives the number of finite integer sequences \((m_1, m_2, \ldots, m_n)\) of length \( n \) such that \( m_1 = 2 \) and \( m_n = 4 \), where \( 1 \leq m_j \leq 4 \) and \( |m_j - m_{j-1}| \leq 1 \) for \( j = 2, 3, \ldots, n-1 \) and \( j = 2, 3, \ldots, n \), respectively. Furthermore, \( B_n \) satisfies, for \( n \geq 5 \), the recurrence relation

\[ B_n = 4B_{n-1} - 3B_{n-2} - 2B_{n-3} + B_{n-4}, \]

with \( B_1 = 1, B_2 = 3, B_3 = 9 \) and \( B_4 = 25 \). In addition we have the following formulas:

\[ B_n = \frac{2}{5} \sum_{k=0}^{4} \sin \left( \frac{2\pi k}{5} \right) \sin \left( \frac{4\pi k}{5} \right) \left( 1 + 2 \cos \left( \frac{\pi k}{5} \right) \right)^{n+1} \]

and

\[ B_n = \frac{1}{2} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} (-1)^k F_{3(n-k)}. \]

The sequence \( \{C_n\} \) is also associated with a one-dimensional random walk, the same one in fact as mentioned above in connection with \( \{B_n\} \), except that now \( m_1 = 1 \). Also, \( C_n \) satisfies the same recurrence relation as \( B_n \), but with the initial conditions \( C_1 = 1, C_2 = 4, C_3 = 13 \) and \( C_4 = 38 \).

REFERENCES


FEBRUARY 2011 55

MSC2010: 05A15, 11B39  

School of Education, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom  
E-mail address: martin.griffiths@manchester.ac.uk