FIBONACCI DIAGONALS

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Abstract. In this paper we obtain a closed-form expression for the sum of the elements lying on the nth diagonal of a Fibonacci triangle. This is achieved by obtaining and then utilizing the ordinary generating functions of two subsequences of the sequence of diagonal sums.

1. Introduction

It is well-known that the nth 'diagonal sum' of Pascal's triangle is equal to F_n; see [4] and [6]. Note that the nth diagonal comprises $\lfloor \frac{n+1}{2} \rfloor$ elements, where $\lfloor x \rfloor$ is the *floor function*, denoting the largest integer not exceeding x.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad \mathbf{1} \\
1 \quad \mathbf{3} \quad 3 \quad 1 \\
\mathbf{1} \quad 4 \quad 6 \quad \mathbf{4} \quad 1 \\
1 \quad 5 \quad \mathbf{10} \quad 10 \quad 5 \quad 1 \\
1 \quad \mathbf{6} \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
\mathbf{1} \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\vdots
\end{array}$$

For example, highlighted in Pascal's triangle above are the 5th and 8th diagonals. We have

$$1+3+1 = 5 = F_5 \qquad \text{and} \qquad 1+6+10+4 = 21 = F_8.$$

In this article we consider the corresponding situation for the Fibonacci triangle $\mathcal{T}$ shown below, which was discussed recently in [3] in connection with an infinite matrix of 0's and 1's that had been constructed from the Zeckendorf representations of the non-negative integers. This triangle also appears as A058071 in Sloane's *On-line Encyclopedia of Integer Sequences* [8]. Furthermore, some of its properties were studied in [5] and [7].

The entry in position p (taken from left to right) of the rth row of $\mathcal{T}$ is equal to F_pF_{r-p+1}. From [2], [3] and [6] we know that the rth row-sum of $\mathcal{T}$ is given by

$$\sum_{p=1}^{r} F_pF_{r-p+1} = \frac{1}{5}\left(rF_{r+1} + 2(r+1)F_r\right).$$

This is the sequence of Fibonacci numbers convolved with themselves, and appears in [8] as A001629.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
2 \quad 1 \quad 2 \\
3 \quad 2 \quad \mathbf{2} \quad 3 \\
5 \quad \mathbf{3} \quad 4 \quad 3 \quad \mathbf{5} \\
\mathbf{8} \quad 5 \quad 6 \quad \mathbf{6} \quad 5 \quad 8 \\
13 \quad 8 \quad \mathbf{10} \quad 9 \quad 10 \quad 8 \quad 13 \\
21 \quad \mathbf{13} \quad 16 \quad 15 \quad 15 \quad 16 \quad 13 \quad 21 \\
\mathbf{34} \quad 21 \quad 26 \quad 24 \quad 25 \quad 24 \quad 26 \quad 21 \quad 34 \\
55 \quad 34 \quad 42 \quad 39 \quad 40 \quad 40 \quad 39 \quad 42 \quad 34 \quad 55 \\
89 \quad 55 \quad 68 \quad 63 \quad 65 \quad 64 \quad 65 \quad 63 \quad 68 \quad 55 \quad 89 \\
\vdots
\end{array}$$

We are interested here in studying the nth diagonal sum A_n of $\mathcal{T}$. The two examples highlighted in $\mathcal{T}$ above show that

$$A_6 = 8 + 3 + 2 = 13 \qquad \text{and} \qquad A_9 = 34 + 13 + 10 + 6 + 5 = 68.$$

It is clear that A_n is equal either to

$$F_1F_n + F_2F_{n-2} + F_3F_{n-4} + \ldots + F_{\frac{n+1}{2}}F_1$$

or to

$$F_1F_n + F_2F_{n-2} + F_3F_{n-4} + \ldots + F_{\frac{n}{2}}F_2,$$

depending on whether n is odd or even, respectively. In fact, more generally it is possible to write

$$\begin{aligned} A_n &= F_1F_n + F_2F_{n-2} + F_3F_{n-4} + \ldots + F_{\lfloor \frac{n+1}{2} \rfloor}F_{n-2\lfloor \frac{n-1}{2} \rfloor} \\ &= \sum_{p=1}^{\lfloor \frac{n+1}{2} \rfloor} F_pF_{n-2(p-1)}. \end{aligned} \tag{1.1}$$

Let us term the above expression a 'semi-stretched convolution'. In this paper we obtain various ordinary generating functions associated with the sequence $\{A_n\}$ and hence a closed-form expression for A_n.

2. Generating Functions

It is easily checked that the first few terms of $\{A_n\}$ are given by:

$$1, 1, 3, 4, 9, 13, 25, 38, 68, 106, \ldots.$$

Our aim in this section is to obtain the ordinary generating function $R(x)$ for $\{A_n\}$,

$$R(x) = A_1x + A_2x^2 + A_3x^3 + \cdots,$$

and two further generating functions associated with $\{A_n\}$.

It is actually very straightforward to calculate $R(x)$. Let $G(x)$ be the ordinary generating function for the Fibonacci numbers. Then from [6] we know that

$$\begin{aligned} G(x) &= F_1 x + F_2 x^2 + F_3 x^3 + \cdots \\ &= \frac{x}{1 - x - x^2} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right), \end{aligned}$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}.$$

Now, noting that

$$R(x) = \left(F_1 x + F_2 x^3 + F_3 x^5 + \cdots\right)\left(F_1 + F_2 x + F_3 x^2 + \cdots\right),$$

we have

$$\begin{aligned} R(x) &= \frac{1}{x} G\left(x^2\right) \cdot \frac{1}{x} G(x) \\ &= \frac{1}{5x^2} \left(\frac{1}{1 - \phi x^2} - \frac{1}{1 - \hat{\phi} x^2} \right) \left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right). \end{aligned}$$

However, it turns out that this is not particularly amenable with respect to finding a closed-form expression for A_n, and the alternative approach we adopt here is to consider $\{A_n\}$ as two interleaved sequences, $\{B_n\}$ and $\{C_n\}$, such that $B_n = A_{2n-1}$ and $C_n = A_{2n}$ for $n \geq 1$. Let us now obtain the ordinary generating functions for $\{B_n\}$ and $\{C_n\}$.

First,

$$\begin{aligned} F_0 + F_2 x^2 + F_4 x^4 + \cdots &= \frac{1}{2} \left(G(x) + G(-x) \right) \\ &= \frac{1}{2\sqrt{5}} \left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} + \frac{1}{1 + \phi x} - \frac{1}{1 + \hat{\phi} x} \right) \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi^2 x^2} - \frac{1}{1 - \hat{\phi}^2 x^2} \right). \end{aligned}$$

Thus it is the case that the generating function $Q_{even}(x)$ for the even-numbered Fibonacci numbers is given by

$$F_0 + F_2 x + F_4 x^2 + \cdots = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right).$$

From this it follows, on using the semi-stretched convolution (1.1), that the generating function

$$V(x) = C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

for $\{C_n\}$ may be expressed as

$$\begin{aligned} V(x) &= G(x) Q_{even}(x) \\ &= \frac{1}{5x} \left(\frac{1}{1 - \phi x} - \frac{1}{1 - \hat{\phi} x} \right) \left(\frac{1}{1 - \phi^2 x} - \frac{1}{1 - \hat{\phi}^2 x} \right). \end{aligned} \tag{2.1}$$

Similarly, since

$$F_1 x + F_3 x^3 + F_5 x^5 + \cdots = \frac{1}{2} \left(G(x) - G(-x) \right),$$

it may be shown that the generating function $Q_{odd}(x)$ for the odd-numbered Fibonacci numbers is given by

$$F_1 + F_3x + F_5x^2 + \cdots = \frac{1}{\sqrt{5}}\left(\frac{\phi}{1-\phi^2x} - \frac{\hat{\phi}}{1-\hat{\phi}^2x}\right),$$

and hence that the generating function

$$U(x) = B_1x + B_2x^2 + B_3x^3 + \cdots$$

for $\{B_n\}$ is

$$U(x) = \frac{1}{5}\left(\frac{1}{1-\phi x} - \frac{1}{1-\hat{\phi}x}\right)\left(\frac{\phi}{1-\phi^2x} - \frac{\hat{\phi}}{1-\hat{\phi}^2x}\right). \tag{2.2}$$

Both $U(x)$ and $V(x)$ will be utilized in Section 3. Incidentally, we may retrieve $R(x)$ from these generating functions as follows:

$$\begin{aligned} R(x) &= \frac{1}{x}U\left(x^2\right) + V\left(x^2\right) \\ &= \frac{1}{5x^2}\left(\frac{1}{1-\phi x^2} - \frac{1}{1-\hat{\phi}x^2}\right)\left(\frac{1+\phi x}{1-\phi^2x^2} - \frac{1+\hat{\phi}x}{1-\hat{\phi}^2x^2}\right) \\ &= \frac{1}{5x^2}\left(\frac{1}{1-\phi x^2} - \frac{1}{1-\hat{\phi}x^2}\right)\left(\frac{1}{1-\phi x} - \frac{1}{1-\hat{\phi}x}\right). \end{aligned}$$

3. A Formula For A_n

Theorem 3.1.

$$A_n = \frac{1}{2}\left(F_{n+3} - F_{2\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n-5}{2}\rfloor}\right).$$

Proof. We start by obtaining a formula for C_n. The right-hand side of (2.1) is multiplied out and then, employing the method of partial fractions, is expressed in the form

$$\frac{1}{5x}\left(\frac{a}{1-\phi^2x} + \frac{b}{1-\hat{\phi}^2x} + \frac{c}{1-\phi x} + \frac{d}{1-\hat{\phi}x}\right)$$

for some $a, b, c, d \in \mathbb{R}$. Subsequently, by expanding each term as a power series in x, comparing coefficients on both sides of (2.1) and using the results

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \quad \text{and} \quad F_n + 2F_{n-1} = \phi^n + \hat{\phi}^n,$$

which may be found in [1] and [6], it can be shown that

$$C_n = \frac{1}{2}\left(F_{2n+3} - F_{n+3}\right).$$

Adopting a similar method with (2.2) leads to the result

$$B_n = \frac{1}{2}\left(F_{2n+2} - F_{n+1}\right).$$

From these expressions for B_n and C_n it does indeed follow that

$$A_n = \frac{1}{2}\left(F_{n+3} - F_{2\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n-5}{2}\rfloor}\right).$$

□

To take an example,

$$\begin{aligned} A_8 &= \frac{1}{2}\left(F_{11} - F_{2\lfloor \frac{8}{2}\rfloor - \lfloor \frac{3}{2}\rfloor}\right) \\ &= \frac{1}{2}\left(F_{11} - F_7\right) \\ &= 38. \end{aligned}$$

We note here that the sequence $\{A_n\}$ does not appear in [8].

4. Further Comments

First, as is noted in [7], the following recurrence relations, valid for $n \geq 1$, follow very easily from the structure of $\mathcal{T}$:

$$A_{2n+1} = A_{2n} + A_{2n-1} + F_{n+1} \quad \text{and} \quad A_{2n+2} = A_{2n+1} + A_{2n},$$

where $A_1 = A_2 = 1$.

Next, it is interesting that both $\{B_n\}$ and $\{C_n\}$ have mathematical lives of their own. We state here, without proof, a number of results associated with these sequences. The interested reader might like to consult [8] in this regard, where $\{B_n\}$ and $\{C_n\}$ appear as A094292 and A056014, respectively.

The sequence $\{B_n\}$ is associated with a particular one-dimensional random walk. Indeed, B_n gives the number of finite integer sequences $(m_1, m_2, \ldots, m_n)$ of length n such that $m_1 = 2$ and $m_n = 4$, where $1 \leq m_j \leq 4$ and $|m_j - m_{j-1}| \leq 1$ for $j = 2, 3, \ldots, n-1$ and $j = 2, 3, \ldots, n$, respectively. Furthermore, B_n satisfies, for $n \geq 5$, the recurrence relation

$$B_n = 4B_{n-1} - 3B_{n-2} - 2B_{n-3} + B_{n-4},$$

with $B_1 = 1$, $B_2 = 3$, $B_3 = 9$ and $B_4 = 25$. In addition we have the following formulas:

$$B_n = \frac{2}{5}\sum_{k=0}^{4} \sin\left(\frac{2\pi k}{5}\right) \sin\left(\frac{4\pi k}{5}\right)\left(1 + 2\cos\left(\frac{\pi k}{5}\right)\right)^{n+1}$$

and

$$B_n = \frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n-k}{k} (-1)^k F_{3(n-k)}.$$

The sequence $\{C_n\}$ is also associated with a one-dimensional random walk, the same one in fact as mentioned above in connection with $\{B_n\}$, except that now $m_1 = 1$. Also, C_n satisfies the same recurrence relation as B_n, but with the initial conditions $C_1 = 1$, $C_2 = 4$, $C_3 = 13$ and $C_4 = 38$.

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MSC2010: 05A15, 11B39

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