SOME BINOMIAL IDENTITIES ASSOCIATED WITH THE GENERALIZED NATURAL NUMBER SEQUENCE

M. A. KHAN AND HARRIS KWONG

Abstract. Define the sequence $\{U_n\}$ as $U_0 = 0$, $U_1 = 1$, and $U_n = pU_{n-1} - U_{n-2}$ for $n \geq 2$. We study $\sum_{h=0}^{n} h^m \binom{n}{h} U_h$ and $\sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} U_h$, and express them in terms of two associated sequences. Special cases of $p = 2, 3$ lead to interesting binomial and Fibonacci identities.

1. Introduction

For any positive integer p, define the sequence $\{U_n\}_{n=0}^{\infty}$ according to

$$U_0 = 0, \quad U_1 = 1, \qquad U_n = pU_{n-1} - U_{n-2}, \quad n \geq 2.$$

The sequence $\{U_n\}$ generates the natural numbers when $p = 2$. For $p > 2$, some familiar properties of the natural numbers can be found in $\{U_n\}$. For instance, using Binet's formula (see below), it is easy to show that

$$\sum_{i=1}^{n} U_{2i-1} = U_n^2,$$

which bears a close resemblance to $\sum_{i=1}^{n} (2i - 1) = n^2$. For this reason we may regard U_n a generalization of the natural numbers.

As in the case of Fibonacci-Lucas sequences, we define the associated sequence $\{V_n\}$ as

$$V_0 = 2, \quad V_1 = p, \qquad V_n = pV_{n-1} - V_{n-2}, \quad n \geq 2.$$

Of particular interest is the case of $p = 3$, in which we find $U_n = F_{2n}$ and $V_n = L_{2n}$.

In this short note, we study two types of summations:

$$\sum_{h=0}^{n} h^m \binom{n}{h} U_h, \qquad \text{and} \qquad \sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} U_h,$$

for some nonnegative integer m. We find relatively simple closed forms for them. Along with an iterative technique and an "exchange" theorem, many combinatorial identities can be obtained. Applications to the special cases of $p = 2, 3$ lead to well-known binomial and Fibonacci identities.

2. Basic Binomial Identities Associated with U_n

To facilitate our discussion, we define two new sequences $\{X_n\}$ and $\{Y_n\}$ according to

$$X_0 = 0, \quad X_1 = 1, \qquad X_n = (p + 2)(X_{n-1} - X_{n-2}), \quad n \geq 2,$$
$$Y_0 = 0, \quad Y_1 = 1, \qquad Y_n = (p - 2)(Y_{n-1} + Y_{n-2}), \quad n \geq 2.$$

Unless otherwise stated, we assume $p > 2$ throughout our exposition.

p	i	0	1	2	3	4	5	6	7	8
2	U_i	0	1	2	3	4	5	6	7	8
	V_i	2	2	2	2	2	2	2	2	2
	X_i	0	1	4	12	32	80	192	448	1024
	Y_i	0	1	0	0	0	0	0	0	0
3	U_i	0	1	3	8	21	55	144	377	987
	V_i	2	3	7	18	47	123	322	843	2207
	X_i	0	1	5	20	75	275	1000	3625	13125
	Y_i	0	1	1	2	3	5	8	13	21
4	U_i	0	1	4	15	56	209	780	2911	10864
	V_i	2	4	14	52	194	724	2702	10084	37634
	X_i	0	1	6	30	144	684	3240	15336	72576
	Y_i	0	1	2	6	16	44	120	328	896

TABLE 1. Values of U_i, V_i, X_i, and Y_i for different values of p.

The Binet forms for U_n, V_n, X_n, and Y_n are given below.

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$V_n = \alpha^n + \beta^n,$$
$$X_n = \frac{(\alpha+1)^n - (\beta+1)^n}{\alpha - \beta},$$
$$Y_n = \frac{(\alpha-1)^n - (\beta-1)^n}{\alpha - \beta},$$

where

$$\alpha = \frac{p + \sqrt{p^2-4}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2-4}}{2}.$$

They yield the following basic results.

Theorem 2.1. *The following identities hold for all $n \geq 0$:*

$$\sum_{h=0}^{n} \binom{n}{h} U_h = X_n, \tag{2.1}$$
$$\sum_{h=0}^{n} h\binom{n}{h} U_h = n(X_n - X_{n-1}), \tag{2.2}$$
$$\sum_{h=0}^{n} (-1)^{n+h}\binom{n}{h} U_h = Y_n, \tag{2.3}$$
$$\sum_{h=0}^{n} (-1)^{n+h}h\binom{n}{h} U_h = n(Y_n + Y_{n-1}). \tag{2.4}$$

Proof. Since

$$\sum_{h=0}^{n} \binom{n}{h}\alpha^h = (1+\alpha)^n \qquad \text{and} \qquad \sum_{h=0}^{n} \binom{n}{h}\beta^h = (1+\beta)^n,$$

(2.1) follows immediately from Binet's formulas. Likewise,

$$\sum_{h=0}^{n}(-1)^{n+h}\binom{n}{h}\alpha^h = (-1)^n\sum_{h=0}^{n}\binom{n}{h}(-\alpha)^h = (-1)^n(1-\alpha)^n = (\alpha-1)^n,$$

and $\sum_{h=0}^{n}(-1)^{n+h}\binom{n}{h}\beta^h = (\beta-1)^n$ give (2.3).

Observe that

$$\sum_{h=0}^{n} h\binom{n}{h}\alpha^h = \alpha\cdot\frac{d}{d\alpha}\left[\sum_{h=0}^{n}\binom{n}{h}\alpha^h\right] = \alpha\cdot\frac{d}{d\alpha}(1+\alpha)^n.$$

Hence,

$$\sum_{h=0}^{n} h\binom{n}{h}\alpha^h = n\alpha(1+\alpha)^{n-1} = n[(1+\alpha)^n - (1+\alpha)^{n-1}]. \tag{2.5}$$

Similarly we obtain

$$\sum_{h=0}^{n} h\binom{n}{h}\beta^h = n[(1+\beta)^n - (1+\beta)^{n-1}], \tag{2.6}$$

which consequently yields (2.2). It is clear that (2.4) can be derived in a similar manner. □

These four identities lay the ground work for many more. Our main tools are the Iterative Technique and the Exchange Theorem, as discussed in the following sections.

3. The Iterative Technique

A scrutiny of (2.5) and (2.6) suggests that an iterative technique for computing $\sum_{h=0}^{n} h^m\binom{n}{h}U_h$ and $\sum_{h=0}^{n}(-1)^{n+h}h^m\binom{n}{h}U_h$, where $m \geq 1$, can be devised. In general, we find

$$\sum_{h=0}^{n} h^m\binom{n}{h}t^h = t\cdot\frac{d}{dt}\left[\sum_{h=0}^{n} h^{m-1}\binom{n}{h}t^h\right].$$

Since $\sum_{h=0}^{n} h^{m-1}\binom{n}{h}t^h$ is a polynomial in $(1+t)$, say,

$$\sum_{h=0}^{n} h^{m-1}\binom{n}{h}t^h = \sum_{k\geq 0} a_k(1+t)^k,$$

it then follows that

$$\sum_{h=0}^{n} h^m\binom{n}{h}t^h = \sum_{k\geq 0} a_k k t(1+t)^{k-1} = \sum_{k\geq 0} a_k k[(1+t)^k - (1+t)^{k-1}].$$

In light of (2.5) and (2.6), define an operator D on X_n as

$$DX_n = n(X_n - X_{n-1}), \qquad \text{for } n \geq 1.$$

It follows from the discussion above that, if $\sum_{h=0}^{n} h^{m-1}\binom{n}{h}U_h$ is of the form $\sum_{k\geq 0} a_k X_k$, then $\sum_{h=0}^{n} h^m\binom{n}{h}U_h = D\left(\sum_{k\geq 0} a_k X_k\right)$. Hence the coefficients a_k can be computed iteratively.

For example, from (2.2), we find

$$\begin{aligned}\sum_{h=0}^{n} h^2\binom{n}{h}U_h &= D\left[\sum_{h=0}^{n} h\binom{n}{h}U_h\right]\\ &= nD(X_n - X_{n-1})\\ &= nDX_n - nDX_{n-1}\\ &= n^2(X_n - X_{n-1}) - n(n-1)[X_{n-1} - X_{n-2}]\\ &= n^2X_n - n(2n-1)X_{n-1} + n(n-1)X_{n-2}. \qquad (3.1)\end{aligned}$$

Note that when $p = 2$, we have $\alpha = \beta = 1$, and

$$\begin{aligned}X_n &= \lim_{p\to 2}\sum_{h=0}^{n}\binom{n}{h}\frac{\alpha^h - \beta^h}{\alpha - \beta}\\ &= \lim_{p\to 2}\sum_{h=0}^{n}\binom{n}{h}\sum_{i=0}^{h-1}\alpha^{h-1-i}\beta^i\\ &= \sum_{h=0}^{n}\binom{n}{h}h\\ &= n\cdot 2^{n-1}.\end{aligned}$$

In addition, since $U_h = h$ when $p = 2$, equation (3.1) becomes

$$\sum_{h=0}^{n} h^3\binom{n}{h} = n^2(n+3)2^{n-3}.$$

This result can be easily verified by noting that

$$\sum_{h=0}^{n} h^3\binom{n}{h}x^h = \left(x\,\frac{d}{dx}\right)^3(1+x)^n,$$

and then by letting $x = 1$.

In an analogous manner, define the operator

$$\Delta Y_n = n(Y_n + Y_{n-1}), \qquad \text{for } n \geq 1.$$

Then one can deduce from (2.4) that

$$\sum_{h=0}^{n}(-1)^{n+h}h^2\binom{n}{h}U_h = n^2Y_n + n(2n-1)Y_{n-1} + n(n-1)Y_{n-2}. \qquad (3.2)$$

In particular, when $p = 3$, $U_h = F_{2h}$, and $Y_n = F_n$, hence

$$\sum_{h=0}^{n}(-1)^{n+h}h^2\binom{n}{h}F_{2h} = n^2F_n + n(2n-1)F_{n-1} + n(n-1)F_{n-2}.$$

Note that, similar to the X_n case, when $p = 2$, we have

$$Y_n = (-1)^n\sum_{h=0}^{n}(-1)^h\binom{n}{h}h = \begin{cases}0 & \text{if } n \neq 1,\\ 1 & \text{if } n = 1,\end{cases}$$

which agrees with the definition of Y_n.

In general, for any $m \geq 1$, the iterative process is summarized in the following theorem.

Theorem 3.1. *For $m \geq 0$, define the polynomials $a_{m,r}(n)$ recursively according to*

$$a_{m,r}(n) = (n-r)a_{m-1,r}(n) - (n-r+1)a_{m-1,r-1}(n), \qquad m \geq 1,$$

with the initial value $a_{0,0}(n) = 1$, and the convention that $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$. Then

$$\sum_{h=0}^{n} h^m \binom{n}{h} U_h = \sum_{r=0}^{m} a_{m,r}(n) X_{n-r}, \tag{3.3}$$

and

$$\sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} U_h = \sum_{r=0}^{m} (-1)^r a_{m,r}(n) Y_{n-r}, \tag{3.4}$$

for all integers $m \geq 0$.

Proof. Recall that

$$\sum_{h=0}^{n} h^m \binom{n}{h} U_h = D\left[\sum_{h=0}^{n} h^{m-1} \binom{n}{h} U_h\right].$$

Hence,

$$\begin{aligned}\sum_{r=0}^{m} a_{m,r}(n) X_{n-r} &= D\left[\sum_{r=0}^{m-1} a_{m-1,r}(n) X_{n-r}\right] \\ &= \sum_{r=0}^{m-1} a_{m-1,r}(n) \cdot (n-r)(X_{n-r} - X_{n-r-1}) \\ &= a_{m-1,0}(n) X_n \\ &\quad + \sum_{r=1}^{m-1} \left[(n-r)a_{m-1,r}(n) - (n-r+1)a_{m-1,r-1}(n)\right] X_{n-r} \\ &\quad - (n-m+1)a_{m-1,m-1}(n) X_{n-m}.\end{aligned}$$

By assuming $a_{m-1,r}(n) = 0$ if $r < 0$ or $r > m-1$, we can write

$$\sum_{r=0}^{m} a_{m,r}(n) X_{n-r} = \sum_{r=0}^{m} \left[(n-r)a_{m-1,r}(n) - (n-r+1)a_{m-1,r-1}(n)\right] X_{n-r}.$$

The recurrence for $a_{m,r}(n)$ follows directly by comparing coefficients. In a similar fashion, define

$$\sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} U_h = \sum_{r=0}^{m} b_{m,r}(n) Y_{n-r}.$$

We find, via Δ,

$$b_{m,r}(n) = (n-r)b_{m-1,r}(n) + (n-r+1)b_{m-1,r-1}(n),$$

and it is clear from the derivation that $b_{m,r}(n) = (-1)^r a_{m,r}(n)$. $\square$

It is obvious from the recurrence that $a_{m,r}(n)$ is a polynomial of degree m for $0 \le r \le m$, and the leading coefficient of $a_{m,r}(n)$ is positive if and only if r is even. In addition, the telescoping nature in the recurrence for $a_{m,r}(n)$ implies

$$\sum_{r=0}^{m} a_{m,r}(n) = 0.$$

Using induction, it is easy to verify that $a_{m,0}(n) = n^m$,

$$b_{m,m}(n) = n(n-1)\cdots(n-m+1) = P(n,m),$$

and

$$b_{m,m-1}(n) = \frac{m(2n-m+1)P(n,m-1)}{2}.$$

These observations help us derive some interesting results.

When $p = 2$, we find $U_h = h$, $Y_1 = 1$, and $Y_n = 0$ if $n \neq 1$. Hence, if we set $m = n-1$, equation (3.4) becomes

$$\sum_{h=0}^{n} (-1)^{n+h} h^n \binom{n}{h} = (-1)^{n-1} a_{n-1,n-1}(n).$$

From

$$a_{n-1,n-1}(n) = (-1)^{n-1} b_{n-1,n-1}(n) = (-1)^{n-1} P(n,n-1),$$

we obtain the remarkable combinatorial identity

$$\sum_{h=0}^{n} (-1)^{n+h} h^n \binom{n}{h} = n!$$

Likewise, by setting $m = n$, we obtain

$$\sum_{h=0}^{n} (-1)^{n+h} h^{n+1} \binom{n}{h} = \frac{n(n+1)!}{2}.$$

We wish to mention one more identity that is not of the type already discussed so far. Note that $\alpha - \beta = \sqrt{p^2-4}$, $\alpha\beta = 1$, $\alpha^2 + 1 = p\alpha$, and $\beta^2 + 1 = p\beta$. We find from the Binet's forms for U_n that

$$\begin{aligned}
(p^2-4)\sum_{h=0}^{n} \binom{n}{h} U_h^2 &= \sum_{h=0}^{n} \binom{n}{h} (\alpha^h - \beta^h)^2 \\
&= \sum_{h=0}^{n} \binom{n}{h} (\alpha^{2h} + \beta^{2h}) - 2\sum_{h=0}^{n} \binom{n}{h} \\
&= (1+\alpha^2)^n + (1+\beta^2)^n - 2 \cdot 2^n \\
&= (p\alpha)^n + (p\beta)^n - 2^{n+1}.
\end{aligned}$$

Therefore,

$$\sum_{h=0}^{n} \binom{n}{h} U_h^2 = \frac{p^n(\alpha^n + \beta^n) - 2^{n+1}}{p^2-4}. \tag{3.5}$$

In particular, for $p > 2$,

$$\sum_{h=0}^{n} \binom{n}{h} U_h^2 = \frac{p^n V_n - 2^{n+1}}{p^2-4}. \tag{3.6}$$

When $p = 3$, $V_n = L_{2n}$ and $U_n = F_{2n}$, hence (3.6) reduces to

$$\sum_{h=0}^{n} \binom{n}{h} F_{2h}^2 = \frac{3^n L_{2n} - 2^{n+1}}{5}.$$

The case of $p = 2$ requires a bit more work, as we need to take the limit as p approaches 2. Since

$$\alpha^n = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} p^{n-j} (p^2 - 4)^{j/2}$$

and

$$\beta^n = \frac{1}{2^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} p^{n-j} (p^2 - 4)^{j/2},$$

we have

$$\alpha^n + \beta^n = \frac{1}{2^{n-1}} \left[p^n + \binom{n}{2} p^{n-2} (p^2 - 4) + \binom{n}{4} p^{n-4} (p^2 - 4)^2 + \cdots \right].$$

Therefore, when $p = 2$, we find (recall that $U_h = h$ if $p = 2$):

$$\begin{aligned}
\sum_{h=0}^{n} \binom{n}{h} h^2 &= \lim_{p \to 2} \frac{p^n(\alpha^n + \beta^n) - 2^{n+1}}{p^2 - 4} \\
&= \lim_{p \to 2} \frac{1}{2^{n-1}} \left[\binom{n}{2} p^{2n-2} + \frac{p^{2n} - 2^{2n}}{p^2 - 4} \right] \\
&= \binom{n}{2} 2^{n-1} + \frac{1}{2^{n-1}} \lim_{p \to 2} \sum_{k=0}^{n-1} (p^2)^{n-1-k} \cdot 4^k \\
&= \binom{n}{2} 2^{n-1} + \frac{1}{2^{n-1}} \cdot n \cdot 4^{n-1} \\
&= n(n+1) 2^{n-2}.
\end{aligned}$$

This result can be verified from

$$\sum_{h=0}^{n} \binom{n}{k} h^2 x^h = \left(x \frac{d}{dx} \right)^2 (1 + x)^n$$

by setting $x = 1$.

4. Exchange Theorem

Identity (3.3) asserts that

$$\sum_{h=0}^{n} h^m \binom{n}{h} U_h = \sum_{r=0}^{m} a_{m,r}(n) X_{n-r}.$$

If we follow the proofs of (2.3) and (3.4), it is not difficult to show that

$$\sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} X_h = \sum_{r=0}^{m} (-1)^r a_{m,r}(n) U_{n-r}.$$

Comparing the two results, we may regard $\{U_n\}$ and $\{X_n\}$ an exchangeable pair of sequences.

In an almost identical manner, it is easy to prove that

$$\sum_{h=0}^{n}\binom{n}{h}Y_h = U_n,$$

from which we deduce that

$$\sum_{h=0}^{m}h^m\binom{n}{h}Y_h = \sum_{r=0}^{m}a_{m,r}(n)U_{n-r}.$$

Compared to (3.4), it is clear that $\{Y_n\}$ and $\{U_n\}$ also form an exchangeable pair of sequences.

More important is the fact that the proofs rely on Binet's formulas only. Hence, identical results hold for similar sequences sharing the same Binet's forms, which in turn depend on the initial values. This means as long as two sequences share the same initial values, and their distinct characteristic roots differ by one, we expect them to form an exchangeable pair of sequences.

For instance, let P_n and Q_n be two sequences with the same initial values $P_0 = Q_0 = 0$, and $P_1 = Q_1 = 1$. Then if λ_1, λ_2 and μ_1, μ_2 are their distinct characteristic roots, such that $\mu_i - \lambda_i = 1$, their Binet's forms will be

$$P_h = \frac{\lambda_1^h - \lambda_2^h}{\lambda_1 - \lambda_2} = \frac{(\mu_1 - 1)^h - (\mu_2 - 1)^h}{\mu_1 - \mu_2},$$

and

$$Q_h = \frac{\mu_1^h - \mu_2^h}{\mu_1 - \mu_2} = \frac{(\lambda_1 - 1)^h - (\lambda_2 - 1)^h}{\lambda_1 - \lambda_2}.$$

The following Exchange Theorem becomes immediate.

Theorem 4.1. *The identities*

$$\sum_{h=0}^{n}h^m\binom{n}{h}P_h = \sum_{r=0}^{m}a_{m,r}(n)Q_{n-r}, \tag{4.1}$$

and

$$\sum_{h=0}^{n}(-1)^{n+h}h^m\binom{n}{h}Q_h = \sum_{r=0}^{m}(-1)^r a_{m,r}(n)P_{n-r} \tag{4.2}$$

hold for all integers $m \geq 0$.

As an illustration let us apply the Exchange Theorem to the identities (2.1) and (2.2). The application of the theorem transforms them to

$$\sum_{h=0}^{n}\binom{n}{h}X_h = U_n,$$

$$\sum_{h=0}^{n}h\binom{n}{h}X_h = n(U_n - U_{n-1}).$$

Now, by noting the fact that for $p = 3$, $U_n = F_{2n}$ and $X_n = F_n$, we can deduce the Fibonacci identities (see [1])

$$\sum_{h=0}^{n} \binom{n}{h} F_h = F_{2n},$$
$$\sum_{h=0}^{n} h \binom{n}{h} F_h = n(F_{2n} - F_{2n-2}) \; = \; nF_{2n-1}.$$

The identity (3.1), on applying Theorem 4.1, yields

$$\sum_{h=0}^{n} h^2 \binom{n}{h} X_h = n^2 U_n - n(2n-1)U_{n-1} + n(n-1)U_{n-2},$$

which reduces to the following Fibonacci identity when $p = 3$:

$$\sum_{h=0}^{n} h^2 \binom{n}{h} F_h = n^2 F_{2n} - n(2n-1)F_{2n-2} + n(n-1)F_{2n-4}.$$

5. Closing Remarks

In this short note, we have demonstrated how the iterative technique and the Exchange Theorem could lead to familiar and some surprising identities. It would be interesting to see if they can be applied to other sequences and what kinds of new identities they may produce. Another research project is to study the properties of the coefficients $a_{m,r}$. Finally, can these techniques be generalized any further?

Acknowledgment

The authors wish to express their appreciation to the anonymous referee for the careful reading and the helpful comments and suggestions that improve the presentation of this note.

References

[1] P. Filipponi, *Some binomial and Fibonacci identities*, The Fibonacci Quarterly, **33.3** (1995), 251–257.

MSC2000: 11B39, 05A19.

c/o A. A. Khan, Regional Office, Indian Overseas Bank, Ashoka Marg, Lucknow, India
E-mail address: manzar57ghazipur@yahoo.co.in

Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063
E-mail address: kwong@fredonia.edu