THIRD AND FOURTH BINOMIAL COEFFICIENTS

ARTHUR T. BENJAMIN AND JACOB N. SCOTT

Abstract. While formulas for the sums of kth binomial coefficients can be shown inductively or algebraically, these proofs give little insight into the combinatorics involved. We prove formulas for the sums of 3rd and 4th binomial coefficients via purely combinatorial arguments.

1. Introduction

In this paper, we present combinatorial proofs of the following two identities.

Theorem 1.1. *For $n \geq 0$,*

$$\sum_{k\geq 0} \binom{n}{3k} = \frac{2^n + m}{3}, \tag{1.1}$$

where m depends on n and is equal to $2, 1, -1, -2, -1, 1$, when n is congruent to $0, 1, 2, 3, 4, 5$ $\pmod 6$, respectively.

Theorem 1.2. *For $n \geq 1$,*

$$\sum_{k\geq 0} \binom{n}{4k} = \frac{2^n + m2^{\lceil n/2 \rceil}}{4}, \tag{1.2}$$

where $m = 2, 1, 0, -1, -2, -1, 0, 1$, when n is congruent, respectively, to $0, 1, 2, 3, 4, 5, 6, 7$ $\pmod 8$.

Both of these identities can be proved by induction or using algebraic methods, or as special cases of the general result [1, 2, 3] that for any integers $0 \leq a < r$ and $n \geq 0$,

$$\sum_{k\geq 0} \binom{n}{a + rk} = \frac{1}{r} \sum_{j=0}^{r-1} \omega^{-ja}(1 + \omega^j)^n,$$

where $\omega = e^{i2\pi/r}$ is a primitive rth root of unity.

But since these are theorems about combinatorial objects, it seems only natural that they should be given combinatorial proofs. In our proof of Theorem 1, we show that among the 2^n subsets of $[n] = \{1, 2, \ldots, n\}$, we can group *almost all* of them into orbits of size three, where in each orbit, the sizes of the three sets are all distinct modulo three. Thus if t_n denotes the number of subsets of $[n]$ with size divisible by three, then t_n should be approximately $2^n/3$.

2. Proofs

Combinatorial Proof of Theorem 1.1.

Suppose X is a subset of $[n]$ that does not contain elements 1 or 2. Then we create the orbit

$$\{X, X \cup \{1\}, X \cup \{1, 2\}\}.$$

Notice that the sizes of the three sets in the orbit are distinct modulo 3. Moreover, every subset of $[n]$ occurs in exactly one orbit of this kind, unless it contains the element 2 but not the element 1. We can extend this rule so that if for some $1 \leq m \leq n/2 - 1$, X contains the numbers $2, 4, \ldots, 2m$, but not the numbers $1, 3, \ldots, 2m-1, 2m+1$, nor the number $2m+2$, then we create the orbit

$$\{X, X \cup \{2m+1\}, X \cup \{2m+1, 2m+2\}\}.$$

Again the sizes of the sets in any orbit are distinct modulo 3, and every subset of $[n]$ appears in exactly one orbit, with the exception of the subset $E_n = \{2, 4, 6, \ldots, n\}$ when n is even, or the two subsets $E_{n-1} = \{2, 4, 6, \ldots, n-1\}$ and $E'_{n-1} = \{2, 4, 6, \ldots, n-1, n\}$ when n is odd.

Thus, if $n \equiv 2$ or $4 \pmod 6$, then $t_n = (2^n - 1)/3$, since $|E_n|$ is not a multiple of three, but if $n \equiv 0 \pmod 6$, then $|E_n|$ is a multiple of three, so the number is increased by one to obtain $t_n = (2^n + 2)/3$. Likewise, if $n \equiv 3 \pmod 6$, then t_n will be $(2^n - 2)/3$, since neither $|E_{n-1}|$ nor $|E'_{n-1}|$ will be multiples of 3, but if $n \equiv 1$ or $5 \pmod 6$, then one of these two numbers is a multiple of three, so we obtain $t_n = (2^n + 1)/3$, as desired. □

We leave it as an exercise for the reader to modify the above proof to show that for $n \geq 0$, $\sum_{k \geq 0} \binom{n}{1+3k}$ and $\sum_{k \geq 0} \binom{n}{2+3k}$ have the same right hand side as Theorem 1, except that m is equal to $-1, 1, 2, 1, -1, -2$ or $-1, -2, -1, 1, 2, 1$, when n is congruent to $0, 1, 2, 3, 4, 5 \pmod 6$, respectively.

Combinatorial Proof of Theorem 1.2.

The left side of this identity counts binary strings of length n where the number of ones is a multiple of four. We call such strings *good*, and let f_n denote the number of good strings of length n. We define the *weight* of a string to be the sum of its entries. Here we place almost all of these 2^n strings into orbits of size four, where the weight of each string in an orbit is distinct modulo four.

Let $X = x_1 x_2 \cdots x_n$ denote a binary string of length n, and let k denote the smallest number for which $x_k = x_{n+1-k}$. If k exists, and $1 \leq k < n/2$, then we create the orbit

$$\{X, X \oplus \{k\} \oplus \{n+1-k\}, X \oplus \{k+1\}, X \oplus \{k\} \oplus \{n+1-k\} \oplus \{k+1\}\},$$

where $X \oplus \{j\}$ is the same string as X with x_j replaced by $x_j + 1 \pmod 2$. For example, the string $X = 1011101110$ has $k = 3$, and $X \oplus \{3\} \oplus \{8\} = 1001101010$. Notice that each object in the orbit has the same value of k and generates the same orbit. If we let w denote the weight of X, then the weights of the elements in the orbit are respectively,

$$w, \quad w+2, \quad w+s, \quad w+2+s \qquad \pmod 4,$$

where s is either 1 or -1. Thus the weights are all distinct modulo four. Let E_n denote the set of "exceptional" strings where k does not exist or (when n is even) is equal to $n/2$. Then since each orbit has exactly one good string, we have

$$f_n = \frac{2^n - |E_n|}{4} + \epsilon_n,$$

where ϵ_n is the number of good exceptional strings. By counting the good exceptional strings, we'll show that the above expression will simplify to

$$f_n = \frac{2^n + m2^{\lceil n/2 \rceil}}{4} \tag{2.1}$$

for the appropriate value of m.

When n is even, we must have $k \leq \frac{n}{2} - 1$, so $|E_n| = 2^{\frac{n}{2}+1}$, since we can freely choose the elements $x_1, \ldots, x_{\frac{n}{2}-1}, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}$, but the remaining elements are forced. Upon "reflection," we see that the weight of such a string must be

$$\frac{n}{2} - 1 + x_{\frac{n}{2}} + x_{\frac{n}{2}+1}.$$

If n is a multiple of 8, then $\frac{n}{2} - 1 \equiv -1 \pmod 4$, and there are two ways to create a good string (by choosing $x_{\frac{n}{2}} = 0, x_{\frac{n}{2}+1} = 1$ or $x_{\frac{n}{2}} = 1, x_{\frac{n}{2}+1} = 0$), hence $\epsilon_n = 2^{\frac{n}{2}}$, and therefore $m = 2$ in expression (2.1). If $n \equiv 2 \pmod 8$, then $\frac{n}{2} - 1 \equiv 0 \pmod 4$, and there is just one way to create a good string (by choosing $x_{\frac{n}{2}} = x_{\frac{n}{2}+1} = 0$), so $\epsilon_n = 2^{\frac{n}{2}-1}$, and therefore $m = 0$ in expression (2.1). By the same line of reasoning, we also get $m = 0$, when $n \equiv 6 \pmod 8$, and when $n \equiv 4 \pmod 8$, we have $\epsilon_n = 0$ and $m = -2$.

When n is odd, then $|E_n| = 2^{\frac{n+1}{2}}$, since the values of $x_1, \ldots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}$ can be freely chosen. The weight of such a string is $\frac{n-1}{2} + x_{\frac{n+1}{2}}$. If $n \equiv 1$ or $7 \pmod 8$, then there is one way to choose $x_{\frac{n+1}{2}}$ to create a good string, resulting in $\epsilon_n = 2^{\frac{n-1}{2}}$ and $m = 1$. Whereas, if $n \equiv 3$ or $5 \pmod 8$, there are no good exceptional strings, resulting in $\epsilon_n = 0$ and $m = -1$. $\square$

Again, we leave it as an exercise to show that for $n \geq 1$, $\sum_{k \geq 0} \binom{n}{1+4k}$, $\sum_{k \geq 0} \binom{n}{2+4k}$, and $\sum_{k \geq 0} \binom{n}{3+4k}$ have the same right-hand side as Theorem 2, except the values of m are replaced by $0, 1, 2, 1, 0, -1, -2, -1$ and $-2, -1, 0, 1, 2, 1, 0, -1$ and $0, -1, -2, -1, 0, 1, 2, 1$, when n is congruent to $0, 1, 2, 3, 4, 5, 6, 7 \pmod 8$, respectively.

3. Acknowlegement

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MSC2010: 05A19

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
E-mail address: benjamin@hmc.edu

Harvey Mudd College, Claremont, CA 91711
E-mail address: jnscott@hmc.edu