POWER SUMS OF PELL AND PELL–LUCAS POLYNOMIALS

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Abstract. Power sums of Pell and Pell–Lucas polynomials are examined. Twelve summation formulas are derived which contain the identities of Ozeki (2009) and Prodinger (2009) as special cases. Furthermore, two general formulas are shown for odd power sums of the unified Pell and Pell–Lucas polynomials.

1. Introduction and Preliminaries

Extending the classical Fibonacci and Lucas numbers, Horadam and Mahon [6] introduced Pell and Pell–Lucas polynomials. They are defined respectively by the recurrence relation

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),$$
$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x);$$

with different initial conditions

$$P_0(x) = 0 \quad \text{and} \quad P_1(x) = 1,$$
$$Q_0(x) = 2 \quad \text{and} \quad Q_1(x) = 2x.$$

They will be shortened as $P_n = P_n(x)$ and $Q_n = Q_n(x)$, which reduce to $P_n(1/2) = F_n$ and $Q_n(1/2) = L_n$, the classical Fibonacci and Lucas numbers. The corresponding generating functions read as

$$\sum_{k=0}^{\infty} P_k(x)y^k = \frac{y}{1-2xy-y^2} = \frac{1}{(\alpha-\beta)(1-y\alpha)} - \frac{1}{(\alpha-\beta)(1-y\beta)},$$
$$\sum_{k=0}^{\infty} Q_k(x)y^k = \frac{2-2xy}{1-2xy-y^2} = \frac{1}{1-y\alpha} + \frac{1}{1-y\beta};$$

which lead to the explicit formulas of Binet forms

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha-\beta} \quad \text{and} \quad Q_n(x) = \alpha^n + \beta^n$$

where for brevity, we employ the following two symbols

$$\alpha := x + \sqrt{x^2+1} \quad \text{and} \quad \beta := x - \sqrt{x^2+1}.$$

It is classically well–known that the mth power sum of the first n natural numbers results in a polynomial of n. Recently, attention has been turned to the similar problem of polynomial representation for power sums of Fibonacci and Lucas numbers, proposed by Melham (cf. [3] and [9]). Ozeki [5, Theorem 2] found the following interesting formula

$$\sum_{k=1}^{n} F_{2k}^{1+2m} = \sum_{i=0}^{m} F_{1+2n}^{1+2i} \sum_{j=i}^{m} \binom{1+2m}{m-j}\binom{i+j+1}{2i+1}\frac{(1+2j)(-5)^{i-m}}{(1+i+j)L_{1+2j}} - \sum_{j=0}^{m} \frac{(-1)^{m-j}}{5^m}\binom{1+2m}{m-j}\frac{F_{1+2j}}{L_{1+2j}}.$$

Prodinger [7] derived 8 formulas for the following power sums of Fibonacci and Lucas numbers

$$\sum_{k=1}^{n} F_{\varepsilon+2k}^{\delta+2m} \quad \text{and} \quad \sum_{k=1}^{n} L_{\varepsilon+2k}^{\delta+2m} \quad \text{where} \quad \varepsilon, \delta \in \{0,1\}.$$

Four of them for odd power sums have recently been unified by Chu and Li [2] via the inversion technique, where two additional parameters are introduced.

Reading carefully Prodinger's paper, we find that his approach can further be employed to investigate the power sums of Pell and Pell–Lucas polynomials

$$\sum_{k=1}^{n} P_{\varepsilon+2k\lambda}^{\delta+2m} \quad \text{and} \quad \sum_{k=1}^{n} Q_{\varepsilon+2k\lambda}^{\delta+2m} \quad \text{where} \quad \lambda \in \mathbb{N}.$$

We shall establish 12 summation formulas, which express the power sums just displayed as polynomials of P_n and Q_n. Six of them may be considered as polynomial extensions of Prodinger's results, while the other remaining six seem new even when considering their reduced cases of Fibonacci and Lucas numbers. What is also remarkable lies in the fact that for each power sum, there exist two polynomial expressions both in P_n and in Q_n.

Throughout the paper, we shall utilize two fundamental lemmas on binomial sums, which contain, for "$y \to 1/x,\ n \to 2n+1$" and "$y \to -1/x,\ n \to 2n$", Prodinger's binomial relations [7, Equations 2.1 and 3.2] as particular cases.

Lemma 1 (Comtet [4, Section 4.9])**.**

$$x^n + y^n = \sum_{0 \le k \le n/2} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k}.$$

Lemma 2 (Carlitz [1, page 23])**.**

$$\frac{x^n - y^n}{x-y} = \sum_{0 \le k < n/2} (-1)^k \binom{n-k-1}{k}(xy)^k(x+y)^{n-2k-1}.$$

Both lemmas can also be found in Swamy [8, Equations 1 and 2] and considered as reduced relations of symmetric functions with two variables. In fact, the second relation can be deduced from the first one as follows. Rewrite the fraction as a finite sum

$$\frac{x^n - y^n}{x-y} = \sum_{i=0}^{n-1} x^i y^{n-i-1} = \sum_{0 \le i < n/2} \frac{x^{n-2i-1} + y^{n-2i-1}}{1+\chi(1+2i=n)}(xy)^i$$

where χ is the logical function defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Applying Lemma 1, we get the double sum expression

$$\frac{x^n - y^n}{x-y} = \sum_{0 \le i+j < n/2} (-1)^j \frac{n-2i-1}{n-2i-j-1}\binom{n-2i-j-1}{j}(xy)^{i+j}(x+y)^{n-2i-2j-1}.$$

The last double sum can be reformulated by letting $i+j=k$

$$\frac{x^n-y^n}{x-y}=\sum_{0\le k<n/2}(xy)^k(x+y)^{n-2k-1}\sum_{j=0}^{k}(-1)^j\frac{n-2k+2j-1}{n-2k+j-1}\binom{n-2k+j-1}{j}.$$

This leads to the right member of the equation displayed in Lemma 2 because the binomial sum with respect to j results in the closed form

$$\sum_{j=0}^{k}(-1)^j\frac{n-2k+2j-1}{n-2k+j-1}\binom{n-2k+j-1}{j}=(-1)^k\binom{n-k-1}{k}$$

which is justified by the telescoping method and the binomial relation

$$\frac{n-2k+2j-1}{n-2k+j-1}\binom{n-2k+j-1}{j}=\binom{n-2k+j-1}{j}+\binom{n-2k+j-2}{j-1}.$$

The rest of the paper will be organized as follows. In the next section, six summation formulas will be derived for power sums of Pell polynomials. Then six similar theorems will be shown for power sums of Pell–Lucas polynomials in the third section. Finally in the fourth section, the paper will end up with illustrating two further formulas for odd powers of $G_n(a,c,x)$, the unified polynomials extending those named by Pell and Pell–Lucas with two extra parameters a and c.

2. Power Sums for Pell Polynomials

In this section, we will investigate power sums of Pell polynomials. According to the binomial theorem, it is trivial to check that

$$\begin{aligned}P_{\varepsilon+2k\lambda}^{\delta+2m} &= \left\{\frac{\alpha^{\varepsilon+2k\lambda}-\beta^{\varepsilon+2k\lambda}}{\alpha-\beta}\right\}^{\delta+2m}\\ &= \sum_{j=-\delta-m}^{m}(-1)^{m-j}\binom{\delta+2m}{m-j}\frac{(\alpha^{\delta+m+j}\beta^{m-j})^{\varepsilon+2k\lambda}}{(\alpha-\beta)^{\delta+2m}}.\end{aligned}$$

Writing the summation domain as

$$\{-\delta-m\le j\le m\}=\{-\delta-m\le j\le -\delta\}\cup\{1-\delta\le j\le m\}$$

and then inverting the summation order, by $j\to -\delta-j$, for the first one, we get the equality

$$\begin{aligned}P_{\varepsilon+2k\lambda}^{\delta+2m} &= \chi(\delta=0)\binom{2m}{m}\frac{(-1)^{m(1+\varepsilon)}}{(\alpha-\beta)^{2m}}+\sum_{j=1-\delta}^{m}(-1)^{m-j}\binom{\delta+2m}{m-j}\\ &\times\left\{\frac{(\alpha^{\delta+m+j}\beta^{m-j})^{\varepsilon+2k\lambda}}{(\alpha-\beta)^{\delta+2m}}+(-1)^\delta\frac{(\alpha^{m-j}\beta^{\delta+m+j})^{\varepsilon+2k\lambda}}{(\alpha-\beta)^{\delta+2m}}\right\}\end{aligned}$$

which can further be simplified into the equation

$$\begin{aligned}P_{\varepsilon+2k\lambda}^{\delta+2m}=\chi(\delta=0)\binom{2m}{m}\frac{(-1)^{m(1+\varepsilon)}}{(\alpha-\beta)^{2m}}+\sum_{j=1-\delta}^{m}\binom{\delta+2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{(\alpha-\beta)^{\delta+2m}}\\ \times\left\{\alpha^{(\delta+2j)(\varepsilon+2k\lambda)}+(-1)^\delta\beta^{(\delta+2j)(\varepsilon+2k\lambda)}\right\}.\end{aligned}$$

Summing over $1 \le k \le n$ for the expression displayed in the last line

$$\begin{aligned}
&\sum_{k=1}^{n}\Big\{\alpha^{(\delta+2j)(\varepsilon+2k\lambda)}+(-1)^{\delta}\beta^{(\delta+2j)(\varepsilon+2k\lambda)}\Big\}\\
&=\alpha^{(\delta+2j)\varepsilon}\frac{\alpha^{2(\delta+2j)\lambda}-\alpha^{2(\delta+2j)(n+1)\lambda}}{1-\alpha^{2(\delta+2j)\lambda}}\\
&\quad+(-1)^{\delta}\beta^{(\delta+2j)\varepsilon}\frac{\beta^{2(\delta+2j)\lambda}-\beta^{2(\delta+2j)(n+1)\lambda}}{1-\beta^{2(\delta+2j)\lambda}}\\
&=\alpha^{(\delta+2j)(\varepsilon-\lambda)}\frac{\alpha^{2(\delta+2j)(n+1)\lambda}-\alpha^{2(\delta+2j)\lambda}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\\
&\quad+\ (-1)^{\delta+\lambda\delta}\beta^{(\delta+2j)(\varepsilon-\lambda)}\frac{\beta^{2(\delta+2j)\lambda}-\beta^{2(\delta+2j)(n+1)\lambda}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\\
&=\frac{\alpha^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}-(-1)^{\delta+\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\\
&\quad-\ \frac{\alpha^{(\delta+2j)(\varepsilon+\lambda)}-(-1)^{\delta+\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}
\end{aligned}$$

we derive the following formula for power sums of Pell polynomials.

Proposition 3 (Reduction formula)**.**

$$\begin{aligned}
\sum_{k=1}^{n}P^{\delta+2m}_{\varepsilon+2k\lambda}=\frac{n(-1)^{m(1+\varepsilon)}}{(\alpha-\beta)^{2m}}\binom{2m}{m}\chi(\delta=0)-\sum_{j=1-\delta}^{m}\frac{(-1)^{(m-j)(1+\varepsilon)}}{(\alpha-\beta)^{\delta+2m}}\binom{\delta+2m}{m-j}\\
\times\ \Big\{\tfrac{\alpha^{(\delta+2j)(\varepsilon+\lambda)}-(-1)^{\delta+\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}-\tfrac{\alpha^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}-(-1)^{\delta+\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\Big\}.
\end{aligned}$$

Based on this proposition, three cases corresponding to "$\delta = 0$", "$\delta = 1,\ \lambda \equiv_2 1$" and "$\delta = 1,\ \lambda \equiv_2 0$" will be examined, where $\lambda \equiv_2 \varepsilon$ stands for the congruence relation $\lambda \equiv \varepsilon \pmod 2$.

2.1. $\delta = 0$**.** In this case, the equation displayed in Proposition 3 becomes

$$\begin{aligned}
\sum_{k=1}^{n}P^{2m}_{\varepsilon+2k\lambda}\quad&=\quad\binom{2m}{m}\frac{n(-1)^{m(1+\varepsilon)}}{(\alpha-\beta)^{2m}}\\
&-\quad\sum_{j=1}^{m}\binom{2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{(\alpha-\beta)^{2m}}\Big\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}-\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{P_{2j\lambda}}\Big\}.
\end{aligned}\tag{1}$$

According to Lemma 2, we get the equality

$$\begin{aligned}
(\alpha-\beta)\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{Q_{\varepsilon+\lambda+2n\lambda}}&=\frac{\alpha^{2j(\varepsilon+\lambda+2n\lambda)}-\beta^{2j(\varepsilon+\lambda+2n\lambda)}}{(\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda})}\\
&=\sum_{i=0}^{j-1}(-1)^{i(\varepsilon+\lambda)}\binom{2j-i-1}{i}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}-\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2j-2i-1}\\
&=\sum_{i=1}^{j}(-1)^{(j-i)(\varepsilon+\lambda)}\binom{i+j-1}{2i-1}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}-\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2i-1}
\end{aligned}$$

and derive the following double sum expression

$$\sum_{k=1}^{n} P_{\varepsilon+2k\lambda}^{2m} = \binom{2m}{m}\frac{n(-1)^{m(1+\varepsilon)}}{4^m(1+x^2)^m} - \sum_{j=1}^{m}\binom{2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m}$$
$$\times\left\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}} - \frac{Q_{\varepsilon+\lambda+2n\lambda}}{P_{2j\lambda}}\sum_{i=1}^{j}(-1)^{(j-i)(\varepsilon+\lambda)}\binom{i+j-1}{2i-1}\frac{P_{\varepsilon+\lambda+2n\lambda}^{2i-1}}{4^{1-i}(1+x^2)^{1-i}}\right\}.$$

This gives rise to the following summation theorem, whose particular case corresponding to $\varepsilon = \lambda = 1$ has been treated by Prodinger [7].

Theorem 4 (Representation in Pell polynomials)**.**

$$\sum_{k=1}^{n} P_{\varepsilon+2k\lambda}^{2m} = \binom{2m}{m}\frac{n(-1)^{m(1+\varepsilon)}}{4^m(1+x^2)^m} - \sum_{j=1}^{m}\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}\binom{2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m}$$
$$+\sum_{i=1}^{m}P_{\varepsilon+\lambda+2n\lambda}^{2i-1}\sum_{j=i}^{m}\binom{2m}{m-j}\binom{i+j-1}{2i-1}\frac{(-1)^{(m-j)(1+\varepsilon)+(j-i)(\varepsilon+\lambda)}Q_{\varepsilon+\lambda+2n\lambda}}{4^{m-i+1}(1+x^2)^{m-i+1}P_{2j\lambda}}.$$

Observe that we also have the equality

$$\begin{aligned}\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{P_{\varepsilon+\lambda+2n\lambda}} &= \frac{\alpha^{2j(\varepsilon+\lambda+2n\lambda)} - \beta^{2j(\varepsilon+\lambda+2n\lambda)}}{\alpha^{\varepsilon+\lambda+2n\lambda} - \beta^{\varepsilon+\lambda+2n\lambda}}\\ &= \sum_{i=0}^{j-1}(-1)^{i(1+\varepsilon+\lambda)}\binom{2j-i-1}{i}\left(\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda}\right)^{2j-2i-1}\\ &= \sum_{i=1}^{j}(-1)^{(j-i)(1+\varepsilon+\lambda)}\binom{i+j-1}{2i-1}\left(\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda}\right)^{2i-1}\end{aligned}$$

and consequently an alternative expression

$$\sum_{k=1}^{n} P_{\varepsilon+2k\lambda}^{2m} = \binom{2m}{m}\frac{n(-1)^{m(1+\varepsilon)}}{4^m(1+x^2)^m} - \sum_{j=1}^{m}\binom{2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m}$$
$$\times\left\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}} - \frac{P_{\varepsilon+\lambda+2n\lambda}}{P_{2j\lambda}}\sum_{i=1}^{j}(-1)^{(j-i)(1+\varepsilon+\lambda)}\binom{i+j-1}{2i-1}Q_{\varepsilon+\lambda+2n\lambda}^{2i-1}\right\}.$$

It can further be stated as another summation formula, whose special case $\varepsilon = 0$ and $\lambda = 1$ can be found in Prodinger [7].

Theorem 5 (Representation in Pell–Lucas polynomials)**.**

$$\sum_{k=1}^{n} P_{\varepsilon+2k\lambda}^{2m} = \binom{2m}{m}\frac{n(-1)^{m(1+\varepsilon)}}{4^m(1+x^2)^m} - \sum_{j=1}^{m}\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}\binom{2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m}$$
$$+\sum_{i=1}^{m}Q_{\varepsilon+\lambda+2n\lambda}^{2i-1}\sum_{j=i}^{m}\binom{2m}{m-j}\binom{i+j-1}{2i-1}\frac{(-1)^{(m-i)(1+\varepsilon)+(j-i)\lambda}P_{\varepsilon+\lambda+2n\lambda}}{4^m(1+x^2)^mP_{2j\lambda}}.$$

2.2. $\delta = 1$ **and** $\lambda \equiv_2 1$**.** Under the replacement $\lambda \to 1 + 2\lambda$, the corresponding equation displayed in Proposition 3 reads as

$$\sum_{k=1}^{n} P^{1+2m}_{\varepsilon+2k+4k\lambda} = \sum_{j=0}^{m} \frac{(-1)^{(m-j)(1+\varepsilon)}}{(\alpha-\beta)^{2m}} \binom{1+2m}{m-j} \left\{ \frac{P_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{Q_{(1+2j)(1+2\lambda)}} - \frac{P_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}} \right\}. \quad (2)$$

According to Lemma 1, we can rewrite the difference

$$\begin{aligned} P_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} &= \frac{\alpha^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} - \beta^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{\alpha-\beta} \\ &= \sum_{i=0}^{j} (-1)^{(j-i)(1+\varepsilon)} \frac{1+2j}{1+2i} \binom{i+j}{2i} (\alpha-\beta)^{2i} P^{1+2i}_{\varepsilon+(1+2n)(1+2\lambda)} \end{aligned}$$

which leads to the following summation theorem.

Theorem 6 (Representation in Pell polynomials)**.**

$$\begin{aligned} \sum_{k=1}^{n} P^{2m+1}_{\varepsilon+2k+4k\lambda} = &- \sum_{j=0}^{m} \frac{P_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}} \binom{1+2m}{m-j} \frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m} \\ &+ \sum_{i=0}^{m} \frac{P^{1+2i}_{1+\varepsilon+2\lambda+2n+4n\lambda}}{4^{m-i}(1+x^2)^{m-i}} \sum_{j=i}^{m} \binom{1+2m}{m-j} \frac{1+2j}{1+2i} \binom{i+j}{2i} \frac{(-1)^{(m-i)(1+\varepsilon)}}{Q_{(1+2j)(1+2\lambda)}}. \end{aligned}$$

This theorem contains two known particular cases. First, Ozeki [5] got the formula corresponding to $\varepsilon = \lambda = 0$. Subsequently, Prodinger [7] derived the case corresponding to $\varepsilon = 1$ and $\lambda = 0$ besides Ozeki's one.

Similarly, by invoking Lemma 2, we also have the equality

$$\begin{aligned} \frac{P_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{P_{1+\varepsilon+2\lambda+2n+4n\lambda}} &= \frac{\alpha^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} - \beta^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{\alpha^{1+\varepsilon+2\lambda+2n+4n\lambda} - \beta^{1+\varepsilon+2\lambda+2n+4n\lambda}} \\ &= \sum_{i=0}^{j} (-1)^{(j-i)\varepsilon} \binom{i+j}{2i} Q^{2i}_{1+\varepsilon+2\lambda+2n+4n\lambda} \end{aligned}$$

which results consequently in another summation formula.

Theorem 7 (Representation in Pell–Lucas polynomials)**.**

$$\sum_{k=1}^{n} P^{2m+1}_{\varepsilon+2k+4k\lambda} = - \sum_{j=0}^{m} \frac{P_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}} \binom{1+2m}{m-j} \frac{(-1)^{(m-j)(1+\varepsilon)}}{4^m(1+x^2)^m}$$

$$+ \frac{P_{1+\varepsilon+2\lambda+2n+4n\lambda}}{4^m(1+x^2)^m} \sum_{i=0}^{m} Q^{2i}_{1+\varepsilon+2\lambda+2n+4n\lambda} \sum_{j=i}^{m} \binom{1+2m}{m-j} \binom{i+j}{2i} \frac{(-1)^{(m-i)\varepsilon+m-j}}{Q_{(1+2j)(1+2\lambda)}}.$$

2.3. $\delta = 1$ **and** $\lambda \equiv_2 0$**.** Under the replacement $\lambda \to 2\lambda$, the corresponding equation displayed in Proposition 3 can be stated as

$$\sum_{k=1}^{n} P^{2m+1}_{\varepsilon+4k\lambda} = \sum_{j=0}^{m} \frac{(-1)^{(m-j)(1+\varepsilon)}}{(\alpha-\beta)^{2+2m}} \binom{1+2m}{m-j} \left\{ \frac{Q_{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{P_{2(1+2j)\lambda}} - \frac{Q_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}} \right\}. \quad (3)$$

In view of Lemma 2, we have the expression

$$\begin{aligned}\frac{Q_{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{Q_{\varepsilon+2\lambda+4n\lambda}} &= \frac{\alpha^{(1+2j)(\varepsilon+2\lambda+4n\lambda)}+\beta^{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{\alpha^{\varepsilon+2\lambda+4n\lambda}+\beta^{\varepsilon+2\lambda+4n\lambda}} \\ &= \sum_{i=0}^{j}(-1)^{(j-i)\varepsilon}\binom{i+j}{2i}(\alpha-\beta)^{2i}P^{2i}_{\varepsilon+2\lambda+4n\lambda}\end{aligned}$$

which leads consequently to the summation theorem.

Theorem 8 (Representation in Pell polynomials)**.**

$$\begin{aligned}\sum_{k=1}^{n}P^{2m+1}_{\varepsilon+4k\lambda} &= -\sum_{j=0}^{m}\frac{Q_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}}\binom{1+2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^{1+m}(1+x^2)^{1+m}} \\ &+\sum_{i=0}^{m}P^{2i}_{\varepsilon+2\lambda+4n\lambda}\sum_{j=i}^{m}\binom{1+2m}{m-j}\binom{i+j}{2i}\frac{(-1)^{(m-i)\varepsilon+m-j}}{\{4(1+x^2)\}^{1+m-i}}\frac{Q_{\varepsilon+2\lambda+4n\lambda}}{P_{2(1+2j)\lambda}}.\end{aligned}$$

Alternatively, applying Lemma 1 gives the expression

$$\begin{aligned}Q_{(1+2j)(\varepsilon+2\lambda+4n\lambda)} &= \alpha^{(1+2j)(\varepsilon+2\lambda+4n\lambda)}+\beta^{(1+2j)(\varepsilon+2\lambda+4n\lambda)} \\ &= \sum_{i=0}^{j}(-1)^{(j-i)(1+\varepsilon)}\frac{1+2j}{1+2i}\binom{i+j}{2i}Q^{1+2i}_{\varepsilon+2\lambda+4n\lambda}\end{aligned}$$

from which we derive another summation formula.

Theorem 9 (Representation in Pell–Lucas polynomials)**.**

$$\begin{aligned}\sum_{k=1}^{n}P^{2m+1}_{\varepsilon+4k\lambda} &= -\sum_{j=0}^{m}\frac{Q_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}}\binom{1+2m}{m-j}\frac{(-1)^{(m-j)(1+\varepsilon)}}{4^{1+m}(1+x^2)^{1+m}} \\ &+\sum_{i=0}^{m}\frac{Q^{1+2i}_{\varepsilon+2\lambda+4n\lambda}}{4^{m+1}(1+x^2)^{m+1}}\sum_{j=i}^{m}\binom{1+2m}{m-j}\frac{1+2j}{1+2i}\binom{i+j}{2i}\frac{(-1)^{(m-i)(1+\varepsilon)}}{P_{2(1+2j)\lambda}}.\end{aligned}$$

3. Power Sums for Pell–Lucas Polynomials

Analogously, the power sums of Pell–Lucas polynomials can be considered. First, it is not difficult to check that

$$\begin{aligned}Q^{\delta+2m}_{\varepsilon+2k\lambda} &= (\alpha^{\varepsilon+2k\lambda}+\beta^{\varepsilon+2k\lambda})^{\delta+2m} \\ &= \sum_{j=-\delta-m}^{m}\binom{\delta+2m}{m-j}(\alpha^{\delta+m+j}\beta^{m-j})^{\varepsilon+2k\lambda}.\end{aligned}$$

Similarly as for $P^{\delta+2m}_{\varepsilon+2k\lambda}$, the last sum can be reformulated as

$$\begin{aligned}Q^{\delta+2m}_{\varepsilon+2k\lambda} = \chi(\delta=0)\binom{2m}{m}(-1)^{m\varepsilon} &+ \sum_{j=1-\delta}^{m}\binom{\delta+2m}{m-j}(-1)^{(m-j)\varepsilon} \\ &\times\Big\{\alpha^{(\delta+2j)(\varepsilon+2k\lambda)}+\beta^{(\delta+2j)(\varepsilon+2k\lambda)}\Big\}.\end{aligned}$$

Summing over $1 \le k \le n$ for the expression displayed in the last line

$$\begin{aligned}
&\sum_{k=1}^{n}\Big\{\alpha^{(\delta+2j)(\varepsilon+2k\lambda)}+\beta^{(\delta+2j)(\varepsilon+2k\lambda)}\Big\}\\
&=\alpha^{(\delta+2j)\varepsilon}\frac{\alpha^{2(\delta+2j)\lambda}-\alpha^{2(\delta+2j)(n+1)\lambda}}{1-\alpha^{2(\delta+2j)\lambda}}\\
&\quad+\beta^{(\delta+2j)\varepsilon}\frac{\beta^{2(\delta+2j)\lambda}-\beta^{2(\delta+2j)(n+1)\lambda}}{1-\beta^{2(\delta+2j)\lambda}}\\
&=\frac{\alpha^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}-(-1)^{\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\\
&\quad-\frac{\alpha^{(\delta+2j)(\varepsilon+\lambda)}-(-1)^{\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}
\end{aligned}$$

we find the following formula for power sums of Pell–Lucas polynomials.

Proposition 10 (Reduction formula)**.**

$$\begin{aligned}
\sum_{k=1}^{n}Q_{\varepsilon+2k\lambda}^{\delta+2m}&=n(-1)^{m\varepsilon}\binom{2m}{m}\chi(\delta=0)-\sum_{j=1-\delta}^{m}(-1)^{(m-j)\varepsilon}\binom{\delta+2m}{m-j}\\
&\times\left\{\frac{\alpha^{(\delta+2j)(\varepsilon+\lambda)}-(-1)^{\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}-\frac{\alpha^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}-(-1)^{\lambda\delta}\beta^{(\delta+2j)(\varepsilon+\lambda+2n\lambda)}}{\alpha^{(\delta+2j)\lambda}-(-1)^{\lambda\delta}\beta^{(\delta+2j)\lambda}}\right\}.
\end{aligned}$$

Now we are ready to examine three cases of the last power sum.

3.1. $\delta=0$**.** In this case, the equation displayed in Proposition 10 reads as

$$\sum_{k=1}^{n}Q_{\varepsilon+2k\lambda}^{2m}\;=\;n(-1)^{m\varepsilon}\binom{2m}{m}-\sum_{j=1}^{m}(-1)^{(m-j)\varepsilon}\binom{2m}{m-j}\left\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}-\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{P_{2j\lambda}}\right\}. \qquad (4)$$

By means of Lemma 2, the last fraction can be written as

$$\begin{aligned}
(\alpha-\beta)\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{Q_{\varepsilon+\lambda+2n\lambda}}&=\frac{\alpha^{2j(\varepsilon+\lambda+2n\lambda)}-\beta^{2j(\varepsilon+\lambda+2n\lambda)}}{\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda}}\\
&=\sum_{i=0}^{j-1}(-1)^{i(\varepsilon+\lambda)}\binom{2j-i-1}{i}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}-\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2j-2i-1}\\
&=\sum_{i=1}^{j}(-1)^{(j-i)(\varepsilon+\lambda)}\binom{i+j-1}{2i-1}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}-\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2i-1}
\end{aligned}$$

which leads to the following double sum expression

$$\begin{aligned}
\sum_{k=1}^{n}Q_{\varepsilon+2k\lambda}^{2m}&=\binom{2m}{m}n(-1)^{m\varepsilon}-\sum_{j=1}^{m}\binom{2m}{m-j}(-1)^{(m-j)\varepsilon}\\
&\times\left\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}-\frac{Q_{\varepsilon+\lambda+2n\lambda}}{P_{2j\lambda}}\sum_{i=1}^{j}\frac{(-1)^{(j-i)(\varepsilon+\lambda)}}{4^{1-i}(1+x^2)^{1-i}}\binom{i+j-1}{2i-1}P_{\varepsilon+\lambda+2n\lambda}^{2i-1}\right\}.
\end{aligned}$$

Interchanging the summation order gives rise to the following theorem which reduces, for $\varepsilon=1$ and $\lambda=1$, to a formula due to Prodinger [7].

Theorem 11 (Representation in Pell polynomials)**.**

$$\sum_{k=1}^{n} Q_{\varepsilon+2k\lambda}^{2m} = (-1)^{m\varepsilon}\binom{2m}{m}n - \sum_{j=1}^{m}(-1)^{(m-j)\varepsilon}\binom{2m}{m-j}\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}$$
$$+\sum_{i=1}^{m} P_{2n\lambda+\varepsilon+\lambda}^{2i-1}\sum_{j=i}^{m}\frac{(-1)^{(m-i)\varepsilon+(j-i)\lambda}}{4^{1-i}(1+x^2)^{1-i}}\binom{2m}{m-j}\binom{i+j-1}{2i-1}\frac{Q_{2n\lambda+\varepsilon+\lambda}}{P_{2j\lambda}}.$$

Alternatively, we can also reformulate the fraction as

$$\frac{P_{2j(\varepsilon+\lambda+2n\lambda)}}{P_{\varepsilon+\lambda+2n\lambda}} = \frac{\alpha^{2j(\varepsilon+\lambda+2n\lambda)} - \beta^{2j(\varepsilon+\lambda+2n\lambda)}}{\alpha^{\varepsilon+\lambda+2n\lambda} - \beta^{\varepsilon+\lambda+2n\lambda}}$$
$$= \sum_{i=0}^{j-1}(-1)^{i(1+\varepsilon+\lambda)}\binom{2j-i-1}{i}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2j-2i-1}$$
$$= \sum_{i=1}^{j}(-1)^{(j-i)(1+\varepsilon+\lambda)}\binom{i+j-1}{2i-1}\Big(\alpha^{\varepsilon+\lambda+2n\lambda}+\beta^{\varepsilon+\lambda+2n\lambda}\Big)^{2i-1}$$

from which we derive another double sum expression

$$\sum_{k=1}^{n} Q_{\varepsilon+2k\lambda}^{2m} = n(-1)^{m\varepsilon}\binom{2m}{m} - \sum_{j=1}^{m}\binom{2m}{m-j}(-1)^{(m-j)\varepsilon}$$
$$\times\left\{\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}} - \frac{P_{\varepsilon+\lambda+2n\lambda}}{P_{2j\lambda}}\sum_{i=1}^{j}(-1)^{(j-i)(1+\varepsilon+\lambda)}\binom{i+j-1}{2i-1}Q_{\varepsilon+\lambda+2n\lambda}^{2i-1}\right\}.$$

This is equivalent to the following summation formula, whose special case corresponding to $\varepsilon = 0$ and $\lambda = 1$ is due to Prodinger [7].

Theorem 12 (Representation in Pell–Lucas polynomials)**.**

$$\sum_{k=1}^{n} Q_{\varepsilon+2k\lambda}^{2m} = n(-1)^{m\varepsilon}\binom{2m}{m} - \sum_{j=1}^{m}(-1)^{(m-j)\varepsilon}\binom{2m}{m-j}\frac{P_{2j(\varepsilon+\lambda)}}{P_{2j\lambda}}$$
$$+\sum_{i=1}^{m} Q_{\varepsilon+\lambda+2n\lambda}^{2i-1}\sum_{j=i}^{m}(-1)^{(j-i)(1+\lambda)+(m-i)\varepsilon}\binom{2m}{m-j}\binom{i+j-1}{2i-1}\frac{P_{\varepsilon+\lambda+2n\lambda}}{P_{2j\lambda}}.$$

3.2. $\delta = 1$ **and** $\lambda \equiv_2 1$**.** Replacing λ by $1+2\lambda$, we may state the corresponding equation displayed in Proposition 10 as

$$\sum_{k=1}^{n} Q_{\varepsilon+2k+4k\lambda}^{1+2m} = \sum_{j=0}^{m}(-1)^{(m-j)\varepsilon}\binom{1+2m}{m-j}\left\{\frac{Q_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{Q_{(1+2j)(1+2\lambda)}} - \frac{Q_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}}\right\}. \quad (5)$$

In view of Lemma 2, the following equality holds

$$\frac{Q_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{Q_{1+\varepsilon+2\lambda+2n+4n\lambda}} = \frac{\alpha^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} + \beta^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)}}{\alpha^{1+\varepsilon+2\lambda+2n+4n\lambda} + \beta^{1+\varepsilon+2\lambda+2n+4n\lambda}}$$
$$= \sum_{i=0}^{j}(-1)^{(j-i)(1+\varepsilon)}\binom{i+j}{2i}(\alpha-\beta)^{2i}P_{1+\varepsilon+2\lambda+2n+4n\lambda}^{2i}$$

from which we establish the following summation theorem.

Theorem 13 (Representation in Pell polynomials)**.**

$$\sum_{k=1}^{n} Q_{\varepsilon+2k+4k\lambda}^{2m+1} = Q_{1+\varepsilon+2\lambda+2n+4n\lambda} \sum_{i=0}^{m} P_{1+\varepsilon+2\lambda+2n+4n\lambda}^{2i} \sum_{j=i}^{m} \binom{1+2m}{m-j} \binom{i+j}{2i}$$
$$\times (-1)^{(m-i)\varepsilon+j-i} \frac{4^i(1+x^2)^i}{Q_{(1+2j)(1+2\lambda)}} - \sum_{j=0}^{m} (-1)^{(m-j)\varepsilon} \binom{1+2m}{m-j} \frac{Q_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}}.$$

Instead, applying Lemma 1, we get another expression

$$\begin{aligned} Q_{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} &= \alpha^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} + \beta^{(1+2j)(1+\varepsilon+2\lambda+2n+4n\lambda)} \\ &= \sum_{i=0}^{j} (-1)^{(j-i)\varepsilon} \frac{1+2j}{1+2i} \binom{i+j}{2i} Q_{1+\varepsilon+2\lambda+2n+4n\lambda}^{1+2i} \end{aligned}$$

which leads to an alternative summation formula.

Theorem 14 (Representation in Pell–Lucas polynomials)**.**

$$\sum_{k=1}^{n} Q_{\varepsilon+2k+4k\lambda}^{2m+1} = -\sum_{j=0}^{m} (-1)^{(m-j)\varepsilon} \binom{1+2m}{m-j} \frac{Q_{(1+2j)(1+\varepsilon+2\lambda)}}{Q_{(1+2j)(1+2\lambda)}}$$
$$+ \sum_{i=0}^{m} Q_{1+\varepsilon+2\lambda+2n+4n\lambda}^{1+2i} \sum_{j=i}^{m} \frac{(-1)^{(m-i)\varepsilon}}{Q_{(1+2j)(1+2\lambda)}} \binom{1+2m}{m-j} \frac{1+2j}{1+2i} \binom{i+j}{2i}.$$

Prodinger [7] found the cases corresponding to $\varepsilon = 0,\ 1$ and $\lambda = 0$.

3.3. $\delta = 1$ **and** $\lambda \equiv_2 0$**.** Under the replacement $\lambda \to 2\lambda$, the corresponding equation displayed in Proposition 10 becomes

$$\sum_{k=1}^{n} Q_{\varepsilon+4k\lambda}^{2m+1} = \sum_{j=0}^{m} (-1)^{(m-j)\varepsilon} \binom{1+2m}{m-j} \left\{ \frac{P_{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{P_{2(1+2j)\lambda}} - \frac{P_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}} \right\}. \tag{6}$$

According to Lemma 1, we have the equality

$$\begin{aligned} P_{(1+2j)(\varepsilon+2\lambda+4n\lambda)} &= \frac{\alpha^{(1+2j)(\varepsilon+2\lambda+4n\lambda)} - \beta^{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{\alpha-\beta} \\ &= \sum_{i=0}^{j} (-1)^{(j-i)\varepsilon} \frac{1+2j}{1+2i} \binom{i+j}{2i} (\alpha-\beta)^{2i} P_{\varepsilon+2\lambda+4n\lambda}^{1+2i} \end{aligned}$$

which yields the following summation theorem.

Theorem 15 (Representation in Pell polynomials)**.**

$$\sum_{k=1}^{n} Q_{\varepsilon+4k\lambda}^{2m+1} = -\sum_{j=0}^{m} (-1)^{(m-j)\varepsilon} \binom{1+2m}{m-j} \frac{P_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}}$$
$$+ \sum_{i=0}^{m} P_{\varepsilon+2\lambda+4n\lambda}^{1+2i} \sum_{j=i}^{m} \frac{(-1)^{(m-i)\varepsilon} 4^i (1+x^2)^i}{P_{2(1+2j)\lambda}} \binom{1+2m}{m-j} \frac{1+2j}{1+2i} \binom{i+j}{2i}.$$

From Lemma 2, we also have the expression

$$\begin{aligned}\frac{P_{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{P_{\varepsilon+2\lambda+4n\lambda}} &= \frac{\alpha^{(1+2j)(\varepsilon+2\lambda+4n\lambda)} - \beta^{(1+2j)(\varepsilon+2\lambda+4n\lambda)}}{\alpha^{\varepsilon+2\lambda+4n\lambda} - \beta^{\varepsilon+2\lambda+4n\lambda}} \\ &= \sum_{i=0}^{j} (-1)^{(j-i)(1+\varepsilon)} \binom{i+j}{2i} Q_{\varepsilon+2\lambda+4n\lambda}^{2i}\end{aligned}$$

which results in another summation formula.

Theorem 16 (Representation in Pell–Lucas polynomials)**.**

$$\begin{aligned}\sum_{k=1}^{n} Q_{\varepsilon+4k\lambda}^{2m+1} = &-\sum_{j=0}^{m} (-1)^{(m-j)\varepsilon} \binom{1+2m}{m-j} \frac{P_{(1+2j)(\varepsilon+2\lambda)}}{P_{2(1+2j)\lambda}} \\ &+ \sum_{i=0}^{m} Q_{\varepsilon+2\lambda+4n\lambda}^{2i} \sum_{j=i}^{m} (-1)^{(m-i)\varepsilon+j-i} \binom{1+2m}{m-j} \binom{i+j}{2i} \frac{P_{\varepsilon+2\lambda+4n\lambda}}{P_{2(1+2j)\lambda}}.\end{aligned}$$

4. Further Formulas for Odd Power Sums

The Pell and Pell–Lucas polynomials can be unified by introducing two extra parameters a and c. They are defined by the recurrence relation

$$G_n(a,c,x) = 2xG_{n-1}(a,c,x) + G_{n-2}(a,c,x) \tag{7}$$

with the initial values being specified by

$$G_0(a,c,x) = a \quad \text{and} \quad G_1(a,c,x) = c. \tag{8}$$

By employing the usual series manipulation (cf. [4, Section 1.13]), it is not difficult to derive the generating function

$$\sum_{k=0}^{\infty} G_k(a,c,x)y^k = \frac{a+(c-2ax)y}{1-2xy-y^2} = \frac{c-2ax+a\alpha}{(\alpha-\beta)(1-y\alpha)} - \frac{c-2ax+a\beta}{(\alpha-\beta)(1-y\beta)}$$

and the explicit formula

$$G_n(a,c,x) = \frac{u\alpha^n - v\beta^n}{\alpha-\beta} \quad \text{with} \quad n \in \mathbb{N}_0 \tag{9}$$

where u and v are two parameters determined by

$$u := c-2ax+a\alpha \quad \text{and} \quad v := c-2ax+a\beta. \tag{10}$$

Then the Pell and Pell–Lucas polynomials are the following particular cases

$$P_n(x) = G_n(0,1,x) \quad \text{and} \quad Q_n(x) = G_n(2,2x,x).$$

When $x = 1/2$, the odd power sums of $G_n(a,c,1/2)$ have recently been investigated by the authors [2]. Following exactly the same procedure presented in that paper, we can further establish the following summation theorems.

Theorem 17 (Unified representation formula)**.**

$$\sum_{k=1}^{n} G_{2k}^{2m+1}(a,c,x) = \sum_{i=0}^{m} \frac{G_{2n+1}^{2i+1}(a,c,x) - c^{1+2i}}{4^{m-i}(1+x^2)^{m-i}} \times \sum_{k=i}^{m} \binom{1+2m}{m-k} \frac{1+2k}{1+2i} \binom{k+i}{2i} \frac{(a^2+2acx-c^2)^{m-i}}{G_{2k+1}(2,2x,x)}.$$

Theorem 18 (Unified representation formula)**.**

$$\sum_{k=1}^{n} G_{2k-1}^{2m+1}(a,c,x) = \sum_{i=0}^{m} \frac{G_{2n}^{2i+1}(a,c,x) - a^{2i+1}}{4^{m-i}(1+x^2)^{m-i}} \times \sum_{k=i}^{m} \binom{1+2m}{m-k} \frac{1+2k}{1+2i} \binom{k+i}{2i} \frac{(c^2-2acx-a^2)^{m-i}}{G_{2k+1}(2,2x,x)}.$$

When "$a = 0$, $c = 1$" and "$a = 2$, $c = 2x$", these formulas agree with those displayed in Theorems 6 and 14 specified with $\varepsilon = 0$, 1 and $\lambda = 0$.

However, we have failed to derive similar results for even power sums.

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