ON THE DISTRIBUTION OF THE EULER FUNCTION WITH FIBONACCI NUMBERS

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Abstract. Here, we show that the distribution function of \( \phi(F_n)/F_n \) is strictly increasing in the interval \([0, 1]\). We also show that the summatory function of \( \phi(F_n)/F_n \) is dense modulo 1.

1. Introduction

Let \( \phi(n) \) be the Euler function of the positive integer \( n \). It is well-known that \( \phi(n)/n \) has a distribution function; that is, for each real number \( u \in [0, 1] \), the density

\[
D_\phi(u) = \lim_{x \to \infty} \frac{1}{x} \# \{ n \leq x : \phi(n)/n \leq u \}
\]

exists. Moreover, \( D_\phi(0) = 0 \), \( D_\phi(1) = 1 \), and the function \( D_\phi(u) \) is strictly increasing and continuous. This was proved by Schoenberg [9]. This result is often said to mark the dawn of probabilistic number theory. In this paper, we look at the distribution function for \( \phi(F_n)/F_n \), where \( (F_m)_{m \geq 1} \) is the Fibonacci sequence given by \( F_1 = 1 \), \( F_2 = 1 \) and \( F_{m+2} = F_{m+1} + F_m \) for all \( m \geq 1 \). There are quite a few papers in the literature treating problems concerning the distribution function for the Euler function with linear recurrence arguments. For example, by modifying appropriately the proof of Theorem 3 in [6], it follows that for each \( u \in [0, 1] \), the density

\[
D_F(u) := \lim_{x \to \infty} \frac{1}{x} \# \{ n \leq x : \phi(F_n)/F_n \leq u \}
\]

exists. Moreover, \( D_F(0) = 0 \), \( D_F(1) = 1 \) and the function \( D_F(u) \) is nondecreasing and continuous. It has not been proved before that \( D_F(u) \) is strictly increasing. A weaker result, namely that \( \{ \phi(F_n)/F_n \}_{n \geq 1} \) is dense in \([0, 1]\) appears in Section 4 of [7]. Here, we prove that this is indeed the case.

Theorem 1.1. The function \( D_F(u) \) is strictly increasing in the interval \([0, 1]\).

A study of multiplicative functions, the distributions of which are strictly increasing, is the topic of [2].

In the previous paper [3], we proved that various means involving the Euler function in the interval \([1, m]\) are dense modulo 1. More specifically, we showed that the sequences of general terms

\[
\sum_{n=1}^{m} \phi(n)/n \mod m
\]

are dense in \([0, 1]\). A study of multiplicative functions, the distributions of which are strictly increasing, is the topic of [2].
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\[ s_n := \sqrt{\sum_{m \leq n} \phi(m)}, \quad a_n := \frac{1}{n} \sum_{m \leq n} \phi(m), \]
\[ g_n := \left( \prod_{m \leq n} \phi(m) \right)^{1/n}, \quad h_n := \frac{n}{\sum_{m \leq n} \frac{1}{\phi(m)}}, \]

respectively, are all dense modulo 1. The first three of them have a linear growth, while the last one has sub-linear growth which is why its study is easier. In the subsequent paper [4], H. Iwaniec together with the first author showed that the sequence \((a_n)_{n \geq 1}\) is in fact uniformly distributed modulo 1. In the paper [5], it was shown that the sequence of general term

\[ c_n := \sum_{m \leq n} \frac{\phi(m^2 + 1)}{m^2 + 1} \quad (1.1) \]

is also dense modulo 1. Here, we treat a sequence similar to \((c_n)_{n \geq 1}\), but where the polynomial input \(m^2 + 1\) is replaced by the \(m\)th Fibonacci number \(F_m\). We have the following result.

**Theorem 1.2.** The sequence of general term

\[ f_n := \sum_{m \leq n} \frac{\phi(F_m)}{F_m} \]

is dense modulo 1.

We point out that it follows as a special case of a result from [8], that

\[ \frac{f_n}{n} = \Gamma_f + O \left( \frac{(\log \log n)^2}{\log n} \right), \]

where

\[ \Gamma_f := \sum_{d \geq 1} \frac{\mu(d)}{dz(d)}, \]

where for a positive integer \(k\) the number \(z(k)\) denotes the smallest positive integer \(m\) such that \(k \mid F_m\), and \(\mu(k)\) is the Möbius function of \(k\). The number \(z(k)\) is sometimes called the order of appearance of \(k\) in the Fibonacci sequence. The constant \(\Gamma_f\) is different from 0; hence, the sequence \((f_n)_{n \geq 1}\) has linear growth. Observe that the sequence \((F_n)_{n \geq 1}\) has exponential growth.

For a positive integer \(\ell\) and a real number \(x > 1\) we put \(\log_{\ell} x\) for the recursively defined function \(\log_1 x := \log x\) and \(\log_{\ell} x := \max\{\log(\log_{\ell-1} x), 1\}\). For a positive integer \(m\) we write \(p(m)\) for the smallest prime factor of \(m\), with the convention that \(p(1) = 1\), and \(\omega(m)\) and \(\tau(m)\) for the number of distinct prime divisors of \(m\) and the total number of divisors of \(m\), respectively. We write \(c_1, c_2, c_3, \text{ etc.}\), for some computable constants which might depend on some other parameters and which are labeled increasingly throughout the paper. We use the Landau symbol \(O\) and the Vinogradov symbol \(\ll\) with their usual meanings.
2. Preliminaries

Let \( \alpha := (1 + \sqrt{5})/2 \) and \( \beta := (1 - \sqrt{5})/2 \) be the characteristic roots of the Fibonacci sequence. Then

\[
F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{for all } m \geq 1.
\]

Let \( m = nj \) for some positive integers \( j \) and \( n \). Put

\[
F_n^{(j)} := \frac{F_{nj}}{F_n} = \frac{\alpha^{nj} - \beta^{nj}}{\alpha^n - \beta^n}.
\]

Then \( F_{nj} = F_j F_n^{(j)} \). Furthermore, it is well-known that if \( p \) is any prime factor dividing both \( F_j \) and \( F_n^{(j)} \), then \( p \) divides both \( n \) and \( F_j \). In particular, if the smallest prime factor of \( n \) exceeds \( F_j \), then \( F_j \) and \( F_n^{(j)} \) are coprime.

The following results appear in [7]. For a positive integer \( m \), let \( \mathcal{P}_m := \{ p : z(p) = m \} \).

**Lemma 2.1.** We have the estimate

\[
\sum_{p \in \mathcal{P}_m} \frac{1}{p} \ll \frac{\log m}{m}.
\]

A better estimate than (2.1) is Lemma 8 in [1], which asserts that the estimate

\[
\sum_{p \in \mathcal{P}_m} \frac{1}{p - 1} \leq \frac{12 + 2 \log \log m}{\phi(m)}
\]

holds for all \( m \geq 2 \).

The working horse of our paper is the following result which is Lemma 8 in [7].

**Lemma 2.2.** Let \( j \in \mathbb{N}, \varepsilon > 0, \gamma \in (0, 1) \) and \( A > 0 \) be all given. Then there exist infinitely many \( n \) with \( p(n) > A \) such that

\[
\frac{\phi(F_n^{(j)})}{F_n^{(j)}} \in (\gamma, \gamma + \varepsilon).
\]

3. The Proofs

3.1. **Proof of Theorem 1.1.** Let \( \gamma \in (0, 1) \) and \( \varepsilon > 0 \) be arbitrary. Use Lemma 2.2 with \( j = 1 \) in order to conclude that there is some \( m \) such that

\[
\frac{\phi(F_m)}{F_m} \in \left( \gamma + \frac{\varepsilon}{2}, \gamma + \varepsilon \right).
\]

Fix such a positive integer \( m \). Choose numbers \( n := m\ell \) for some positive integer \( \ell \). We let \( y > m \) be a parameter to be made more precise later and look only at the positive integers \( \ell \) which are free of primes \( p \leq y \). The proportion of them is

\[
\prod_{p \leq y} \left( 1 - \frac{1}{p} \right) > \frac{c_2}{\log y},
\]

where, by Mertens’ Theorem, we can take \( c_2 := e^{-\gamma}/2 \) provided that \( y \) is sufficiently large. Since \( p(\ell) > m \), it follows that \( \ell \) and \( m \) are coprime so that

\[
\frac{\phi(F_n)}{F_n} = \frac{\phi(F_m)}{F_m} \frac{\phi(F_{\ell}^{(m)})}{F_{\ell}^{(m)}}.
\]
It suffices to show that there is a positive proportion of such $\ell$ with the property that
\[
1 - \varepsilon \leq \frac{\phi(F_{\ell}^{(m)})}{F_{\ell}^{(m)}},
\]  
(3.4)
because then by estimates (3.1), (3.2), and (3.4), it will follow easily that there exists a positive proportion of $n$ such that
\[
\phi(F_n) \in (\gamma, \gamma + \varepsilon),
\]  
showing that in fact $D_F(\gamma + \varepsilon) > D_F(\gamma)$. Since this is true for arbitrary $\gamma \in (0, 1)$ and $\varepsilon > 0$, we conclude the proof of this theorem.

To construct the positive proportion of positive integers $\ell$ with the desired property (3.4), we proceed as follows. Note first that
\[
\frac{\phi(F_{\ell}^{(m)})}{F_{\ell}^{(m)}} = \prod_{\substack{z(p)|m \ell \quad \text{or} \quad z(p)|m \ell
}} \left(1 - \frac{1}{p}\right) = \prod_{d_1|\ell} \prod_{d_2|m \quad z(p) = d_1 d_2} \prod_{d_1 > 1} \left(1 - \frac{1}{p}\right)
\]
\[
= \exp \left(\sum_{d_1|\ell} \sum_{d_2|m} \sum_{p \in \mathcal{P}} \frac{1}{p} + O \left(\sum_{z(p) > y} \frac{1}{p^2}\right)\right)
\]
\[
> \exp \left(-c_3 \sum_{d_1|\ell} \sum_{d_2|m \quad d_1 > 1} \log \frac{d_1 d_2}{d_1 d_2} + O \left(\frac{1}{y}\right)\right). 
\]
(3.5)

In the above estimates, we used Lemma 2.1 together with the fact that since $p \equiv \pm 1 \pmod{z(p)}$ holds for all primes $p \neq 5$, then if $z(p) = d_1 d_2$ for some divisor $d_1 > 1$ of $\ell$ and some divisor $d_2$ of $m$, it follows that $d_1 > y$, so $p \geq z(p) - 1 \geq d_1 - 1 > y - 1$. Let $c_4$ be the implicit constant in the Landau symbol in (3.5). Since $d_1 > y > m \geq d_2$, we have
\[
\frac{\log(d_1 d_2)}{d_1 d_2} \leq \frac{\log(d_1^2)}{d_1} = \frac{2 \log(d_1)}{d_1},
\]
so
\[
\sum_{d_1|\ell} \sum_{d_2|m \quad d_1 > 1} \frac{\log(d_1 d_2)}{d_1 d_2} \leq 2 \tau(m) \sum_{d|\ell \quad d > 1} \log d.
\]
Observing that $\log 1 = 0$, we get with $c_5 := 2 \tau(m) c_3 + c_4$ and
\[
S_\ell := \sum_{d|\ell} \frac{\log d}{d},
\]
that the inequality
\[
\frac{\phi(F_{\ell}^{(m)})}{F_{\ell}^{(m)}} > \exp(-c_5 S_\ell) 
\]
holds.
Now observe that
\[ \sum_{\ell \leq x} S_{\ell} = \sum_{\ell \leq x} \sum_{d \mid \ell} \frac{\log d}{d} = \sum_{\ell \leq x} \frac{\log d}{d} \sum_{d \leq \ell, (d \mid \ell)} 1 \]
\[ = \sum_{\ell \leq x} \frac{\log d}{d} \left\lfloor \frac{x}{d} \right\rfloor \leq x \sum_{d \geq y} \frac{\log d}{d^2} \leq x \int_{y-1}^{\infty} \frac{(\log t)dt}{t^2} < \frac{c_6 x \log y}{y}. \]

Thus, a first moment argument shows that the set of \( \ell \leq x \) such that \( S_{\ell} \geq 1/\sqrt{y} \) is of cardinality \( < c_6 (\log y)/\sqrt{y} \). If \( y \) is so large that
\[ \frac{c_2}{\log y} - \frac{c_6 (\log y)}{\sqrt{y}} > 0, \]
it then follows, by estimate (3.2), that a positive proportion of our \( \ell \) have the property that \( S_{\ell} < 1/\sqrt{y} \). For such \( \ell \), we have
\[ \frac{\phi(F_{\ell}^{(m)})}{F_{\ell}^{(m)}} > \exp(-c_5 S_{\ell}) > 1 - c_5 S_{\ell} > 1 - \frac{c_5}{\sqrt{y}} > 1 - \frac{\varepsilon}{4} \]
provided also that \( y > (4c_5/\varepsilon)^2 \). This is what we wanted to prove.

4. The Proof of Theorem 1.2

Here we proceed as in [3], or as in Section 5 of [7]. We first let \( k > 1 \) be an integer. Then we choose \( \varepsilon > 0 \) small enough with respect to \( k \) in a way that will be made more precise later.

Put \( K := k!^2 \). We let \( n \) be such that \( n \equiv 0 \pmod{K} \). Write \( n := K \ell \). Now substitute for \( j = 1, \ldots, k \),
\[ n + j = K \ell + j = j \left( K \ell + 1 \right) =: jn_j, \quad \text{for} \quad j = 1, \ldots, k. \]

Assume that the smallest prime factor of \( n_1 n_2, \ldots, n_k \) is \( > F_k \). Then, \( F_{n+j} = F_j F_{n_j}^{(j)} \) and the two factors on the right are coprime. Hence,
\[ \frac{\phi(F_{n+j})}{F_{n+j}^{(m)}} = \frac{\phi(F_j)}{F_j} \frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}}. \]

For simplicity, write \( C_j := \phi(F_j)/F_j \). We want to choose \( n \) such that the inclusion
\[ \frac{\phi(F_{n+j})}{F_{n+j}} \in \left( \frac{\varepsilon}{2}, \varepsilon \right) \]
holds. This is equivalent to
\[ \frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}} \in \left( \frac{\varepsilon}{2C_j}, \frac{\varepsilon}{C_j} \right). \]

But \( C_j \) is sub-unitary, so the condition we need is that \( \varepsilon \) is sufficiently small. Clearly, \( C_1 = C_2 = 1 \) and \( C_3 = 1/2 \). Since the inequality
\[ \frac{\phi(m)}{m} \geq \frac{c_7}{\log \log m} \]

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holds for all \( m > 3 \) with some positive constant \( c_7 \), it follows that if \( j \geq 4 \) (hence, \( F_j \geq 3 \)), then

\[
C_j = \frac{\phi(F_j)}{F_j} \geq \frac{c_7}{\log \log F_j} > \frac{c_7}{\log j} \geq \frac{c_7}{\log k}.
\]

holds for all \( 4 \leq j \leq k \), where we used the fact that the inequality \( F_j < e^j \) holds for all \( j \geq 1 \).

Now taking \( \varepsilon := 1/(\log k)^2 \), we then get that

\[
\frac{\varepsilon}{C_j} \leq \frac{1}{c_7 \log k}
\]

and the quantity on the right above is \( < 1 \) for all sufficiently large \( k \).

Putting \( \gamma_j := 2\varepsilon/(3C_j) \) and taking \( \varepsilon_1 := \min\{\varepsilon/(12C_j)\} \), Lemma 2.2 applied with \( \varepsilon \) replaced by \( \varepsilon_1 \) successively with \( j = 1, \ldots, k \) implies that there exist \( m_1, \ldots, m_k \) which are pairwise coprime and such that the smallest prime factor of \( m_1m_2 \cdots m_k \) exceeds \( F_k \) and such that furthermore

\[
\frac{\phi(F_{m_j}^{(j)})}{F_{m_j}^{(j)}} \in (\gamma_j, \gamma_j + \varepsilon_1) \subseteq \left( \frac{2\varepsilon}{3C_j}, \frac{\varepsilon}{C_j} \right) \quad \text{for } j = 1, \ldots, k. \tag{4.3}
\]

Indeed, for \( j = 1 \), we choose \( \gamma := \gamma_1 \) and \( A := F_k \) and invoke Lemma 2.2 with \( \varepsilon \) replaced by \( \varepsilon_1 \) to find \( m_1 \) such that containment (4.3) holds with \( j = 1 \). Next, we choose \( j = 2, A := F_km_1 \), and \( \gamma := \gamma_2 \) and apply again Lemma 2.2 with \( \varepsilon \) replaced by \( \varepsilon_1 \) to get \( m_2 \) whose smallest prime factor exceeds both \( F_k \) and \( m_1 \) such that containment (4.3) holds for \( j = 2 \). So \( m_2 \) is coprime to \( m_1 \). Inductively, we construct \( m_1, \ldots, m_k \) with the desired distributional and arithmetic properties. Now to get to \( n \) satisfying (4.2), we first use the Chinese Remainder Lemma to solve \( n_j \equiv 0 \pmod{m_j} \), which is possible since \( m_1, \ldots, m_k \) are coprime in pairs. This puts \( \ell \) into an arithmetic progression \( N \) modulo \( M := m_1 \cdots m_k \). Writing \( \ell := N + M\lambda \), we get that

\[
(n + 1)(n + 2) \cdots (n + k) = k!m_1m_2 \cdots m_k Q(\lambda),
\]

where \( Q(X) \in \mathbb{Z}[X] \) is a linear polynomial with simple roots which factors in linear factors over the integers. By the sieve, there exist constants \( c_8 \) and \( c_9 \) such that for large \( x \), there are \( > c_7x/(\log x)^k \) values of \( \lambda \leq x \) with the property that the smallest prime factor of \( Q(\lambda) \) exceeds \( x^{c_9} \). Thus, if we write \( n_j := m_j \ell_j \), then \( \omega(\ell_j) < c_{10} \). For large \( x \) (say, so large that \( x^{c_9} > F_{km_j} \)), we have

\[
\frac{\phi(F_{m_j}^{(j)})}{F_{m_j}^{(j)}} = \frac{\phi(F_{m_j}^{(j)})}{F_{m_j}^{(j)}} \frac{\phi(F_{\ell_j}^{(jm_j)})}{F_{\ell_j}^{(jm_j)}}.
\]

The primes dividing \( F_{\ell_j}^{(jm_j)} \) all have indices of appearance divisible by a prime factor of \( \ell_j \), which is larger than \( x^{c_9} \). Thus, by Lemma 2.1, we get easily via an argument similar to the one used in the proof of Theorem 1.1, that

\[
\frac{\phi(F_{\ell_j}^{(jm_j)})}{F_{\ell_j}^{(jm_j)}} = \exp \left( O \left( \frac{\log x}{x^{c_9}} \right) \right), \tag{4.4}
\]

where the constant implied by the last \( O \) symbol above depends on \( jm_j \) (in fact, for large \( x \) it depends only on \( \tau(jm_j) \)). At any rate, as \( x \) tends to infinity, the right–hand side in estimate (4.4) above tends to \( 1 \), so if \( x \) is sufficiently large then this quantity will become larger than
1 − ε₁. With estimate (4.3), we then get that

\[ \frac{\phi(F_{n+j})}{F_{n+j}} \in (\gamma_1 - \varepsilon_1, \gamma_1 + \varepsilon_1) \subseteq \left( \frac{\varepsilon}{2C_j}, \frac{\varepsilon}{C_j} \right), \]

which is exactly the containment (4.2) for \( j \).

Now having done the hardest part, the proof is not that far off. Namely, it is clear that with a large \( n \) satisfying the containments (4.2) for \( j = 1, \ldots, k \), we have

\[ f_{n+j} - f_{n+j-1} = \frac{\phi(F_{n+j})}{F_{n+j}} = c_j \frac{\phi(F_{n+j})}{F_{n+j}} \in (0, \varepsilon), \]

while the sum of the above numbers for \( j = 1, \ldots, k \) is

\[ f_{n+k} - f_n = \sum_{j=1}^{k} \frac{\phi(F_{n+j})}{F_{n+j}} > \frac{k\varepsilon}{2} = \frac{k}{2(\log k)^2} > 1, \]

for \( k \) sufficiently large, from where one deduces easily that \( (f_n)_{n \geq 1} \) is indeed dense modulo 1.

### 5. Concluding Remarks

The method of proofs of both results extends easily to other sequences. For example, it extends to the function \( \nu(F_n) \), where \( \nu \) is a multiplicative function, not necessarily strongly multiplicative, with values in \((0,1]\) such that for some positive constant \( c \) we have

\[ \nu(p^a) = 1 - \frac{c}{p} + O\left(\frac{1}{p^2}\right) \quad \text{for all primes} \quad p \geq 2 \quad \text{and integers} \quad a \geq 1. \]

For example, we can replace \( \phi(F_m)/F_m \) with \( F_m/\sigma(F_m) \), where \( \sigma(m) \) is the sum of the divisors of \( m \), and keep the conclusions about the monotonicity of the distribution function and the density modulo 1 of the summatory function. It also applies when the sequence of Fibonacci numbers \( (F_m)_{m \geq 1} \) is replaced with the sequence of Mersenne numbers \( (2^m - 1)_{m \geq 1} \), or some other Lucas sequences satisfying some mild technical conditions.

### References


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