INVERTING A FINITE SERIES WITH CONSTANT COEFFICIENTS

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Abstract. The purpose of this paper is to solve for $f(n)$ where

$$g_r(n) = \sum_{k=0}^{r} a_k f(n-k), \tag{$*$}$$

where $f(n) = 0$ if $n < 0$, and $\{a_0, a_1, \ldots\}$ are constants. The main results are a recursive formula and an explicit formula for the inversion of the series defined by (*).

1. Introduction

In [1], Gould studied the sequence $(g(n))_{n=0}^{\infty}$, where $(g(n))_{n=0}^{\infty}$ was defined by

$$g(n) = \sum_{k=0}^{n} f(n-k), \tag{1.1}$$

where $f(n) = 0$ for $n < 0$, with $1 \leq r \leq n$. Gould primarily investigated how to solve Equation (1.1) for $f(n)$. Jamie Simpson, in writing his review for MathSciNet [2], was surprised that no one had previously studied the inversion of Equation (1.1). In this same review, Simpson mentioned that a natural generalization of Equation (1.1) would be

$$g_r(n) = \sum_{k=0}^{r} a_k f(n-k), \tag{1.2}$$

where $f(n) = 0$ if $n < 0$, and $\{a_0, a_1, \ldots\}$ are constants. This paper studies the inversion of Equation (1.2). The main results are Theorems 1.1 and 1.2. These theorems show how to solve Equation (1.2) for $f(n)$ when $r \geq 1$. In particular, Theorem 1.1 provides a simple recursive methodology for forming the inversion, while Theorem 1.2 provides an explicit formula for the initial stage of the recurrence provided by Theorem 1.1. The explicit formula in Theorem 1.2 utilizes multinomial coefficients and generalizes Gould's observation [1] that

$$f(n) = \sum_{k=0}^{n} g(n-k) \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^{k-j} \binom{k-j}{j}, \tag{1.3}$$

whenever $g(n) = f(n) + f(n-1) + f(n-2)$, with $f(n) = 0$ for $n < 0$.

Section 3 provides an alternative explanation why, when $a_0 = a_1 = \cdots = a_r = 1$, the values of g which have nonzero coefficients in the inversion obey the third order recurrence $B_n = B_{n-1} + B_{n-2} - B_{n-3}$. Finally, Section 4 extends Simpson's generalization to the context of coefficients dependent on r as well as k.

2. Inverting $g_r(n) = \sum_{k=0}^{r} a_k f(n-k)$

In this section, we let $(f(n))_{n=0}^{\infty}$ be an arbitrary sequence. We assume r is a fixed positive integer. We define the sequence $(g_r(n))_{n=0}^{\infty}$ by the recurrence

$$g_r(n) = \sum_{k=0}^{r} a_k f(n-k), \tag{2.1}$$

where $f(n) = 0$ if $n < 0$, and $\{a_k\}_{k=0}^{\infty}$ are given constants.

It is natural to ask whether the inversion of Equation (2.1) has a "nice" format. The answer to this query, as we shall see, is yes. For example, if $r = 1$, Equation (2.1) implies $g_1(n) = a_0 f(n) + a_1 f(n-1)$. By induction, we easily show that

$$a_0^{n+1} f(n) = \sum_{k=0}^{n} (-1)^{n-k} a_0^k a_1^{n-k} g_1(k). \tag{2.2}$$

Now let $r = 2$. Then, Equation (2.1) implies $g_2(n) = a_0 f(n) + a_1 f(n-1) + a_2 f(n-2)$. Inverting this expression for small values of n, namely for $0 \le n \le 4$, we find that

$$\begin{aligned}
a_0 f(0) &= g_2(0)\\
a_0^2 f(1) &= a_0 g_2(1) - a_1 g_2(0)\\
a_0^3 f(2) &= a_0^2 g_2(2) - a_0 a_1 g_2(1) + (a_1^2 - a_0 a_2) g_2(0)\\
a_0^4 f(3) &= a_0^3 g_2(3) - a_0^2 a_1 g_2(2) + (a_0 a_1^2 - a_0^2 a_2) g_2(1) + (2a_0 a_1 a_2 - a_1^3) g_2(0)\\
a_0^5 f(4) &= a_0^4 g_2(4) - a_0^3 a_1 g_2(3) + (a_0^2 a_1^2 - a_0^3 a_2) g_2(2) + (2a_0 a_1 a_2 - a_0 a_1^3) g_2(1)\\
&\quad + (a_1^4 - 3a_1^2 a_2 a_0 + a_0^2 a_2^2) g_2(0).
\end{aligned}$$

By inspecting the five previous equations, we notice that the coefficient of $g_2(k)$ is a polynomial in a_0, a_1, and a_2. We will call this coefficient $P_k^n(a_0, a_1, a_2) = P_k^n$. The P_k^n satisfy the simple recurrence given in Lemma 2.1.

Lemma 2.1. *Let $(f(n))_{n=0}^{\infty}$ be an arbitrary sequence. Let $(g_2(n))_{n=0}^{\infty}$ be defined by the recurrence $g_2(n) = a_0 f(n) + a_1 f(n-1) + a_2 f(n-2)$, where $f(n) = 0$ if $n < 0$, and $\{a_0, a_1, a_2\}$ are given constants. Then for $n \ge 0$,*

$$a_0^{n+1} f(n) = \sum_{k=0}^{n} P_k^n g_2(k), \tag{2.3}$$

where, for $k > 0$,

$$P_k^n = a_0 P_{k-1}^{n-1}, \tag{2.4}$$

and if $k = 0$,

$$P_0^n = -a_1 P_0^{n-1} - a_2 P_1^{n-1}, \quad n \ge 1, \tag{2.5}$$

with

$$P_0^0 = 1, \quad P_k^n = 0, \text{ if } k > n. \tag{2.6}$$

Proof of Lemma 2.1. We will prove Lemma 2.1 by induction on n in Equation (2.3). First note that Equation (2.3) is obviously true if $n = 0$. Now assume Equation (2.3) is true for any $m < n$. We take the recurrence $g_2(n) = a_0 f(n) + a_1 f(n-1) + a_2 f(n-2)$, and multiply both sides by a_0^n. This gives us

$$a_0^n g_2(n) = a_0^{n+1} f(n) + a_1 a_0^n f(n-1) + a_2 a_0^n f(n-2).$$

By the induction hypothesis, the previous line becomes

$$\begin{aligned} a_0^n g_2(n) &= a_0^{n+1} f(n) + a_1 \sum_{k=0}^{n-1} P_k^{n-1} g_2(k) + a_2 a_0 \sum_{k=0}^{n-2} P_k^{n-2} g_2(k) \\ &= a_0^{n+1} f(n) + a_1 P_{n-1}^{n-1} g_2(n-1) + \sum_{k=0}^{n-2} \left[a_1 P_k^{n-1} + a_0 a_2 P_k^{n-2} \right] g_2(k). \end{aligned}$$

Since we want $a_0^{n+1} f(n) = \sum_{k=0}^{n} P_k^n g_2(k)$, the last equality implies

$$P_k^n = -a_1 P_k^{n-1} - a_0 a_2 P_k^{n-2}, k \neq n \tag{2.7}$$

$$P_n^n = a_0^n. \tag{2.8}$$

Using Equation (2.7), we are now in a position to prove Equation (2.4). We use induction on n. First note $P_1^1 = a_0 = a_0 \cdot 1 = a_0 \cdot P_0^0$. Also, Equation (2.8) clearly implies $P_n^n = a_0 P_{n-1}^{n-1}$. If $k \neq n$, we have

$$\begin{aligned} P_k^n &= -a_1 P_k^{n-1} - a_0 a_2 P_k^{n-2}, && \text{by Equation (2.7)} \\ &= -a_1 a_0 P_{k-1}^{n-2} - a_0^2 a_2 P_{k-1}^{n-3}, && \text{by induction hypothesis} \\ &= a_0 \left[-a_1 P_{k-1}^{n-2} - a_0 a_2 P_{k-1}^{n-3} \right] \\ &= a_0 P_{k-1}^{n-1}, && \text{by Equation (2.7),} \end{aligned}$$

which is identically Equation (2.4).

It remains to prove Equation (2.5). By Equation (1.7), we have

$$\begin{aligned} P_0^n &= -a_1 P_0^{n-1} - a_2 a_0 P_0^{n-2} \\ &= -a_1 P_0^{n-1} - a_2 P_1^{n-1}, && \text{by Equation (2.4),} \end{aligned}$$

which is Equation (2.5). □

Lemma 1.1 provides a simple recursive way of determining the inversion procedure. Notice that both Equations (2.4) and (2.5) depend on P_0^n. Thus, it would be beneficial to find an explicit formula for P_0^n. Such a formula is given in Lemma 2.2.

Lemma 2.2. *Let P_0^n be as defined by Lemma 2.1. Then,*

$$P_0^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-k}{k} a_1^{n-2k} a_0^k a_2^k. \tag{2.9}$$

Proof of Lemma 2.2. First note that Equation (2.9) implies that $P_0^0 = 1$, which clearly agrees with Equation (2.6). We will use induction on n and assume that Equation (2.9) holds for all $m < n$. Then, by Equation (2.7) and the induction hypothesis, we have

$$\begin{aligned} P_0^n &= -a_1 P_0^{n-1} - a_0 a_2 P_0^{n-2} \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-k} \binom{n-1-k}{k} a_1^{n-2k} a_0^k a_2^k + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^{n-1-k} \binom{n-2-k}{k} a_1^{n-2-2k} a_0^{k+1} a_2^{k+1} \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-k} \binom{n-1-k}{k} a_1^{n-2k} a_0^k a_2^k + \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor + 1} (-1)^{n-k} \binom{n-1-k}{k-1} a_1^{n-2k} a_0^k a_2^k. \end{aligned} \tag{2.10}$$

We must now simplify Equation (2.10). This simplification has two cases. First, if n is even, note that $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor + 1$ and $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$. Equation (2.10) becomes

$$\begin{aligned} P_0^n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{n-k} \binom{n-1-k}{k} a_1^{n-2k} a_0^k a_2^k + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-1-k}{k-1} a_1^{n-2k} a_0^k a_2^k \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{n-k} \left[\binom{n-1-k}{k} + \binom{n-1-k}{k-1} \right] a_1^{n-2k} a_0^k a_2^k + (-1)^n a_1^n + (-1)^{\lfloor \frac{n}{2} \rfloor} a_0^{\lfloor \frac{n}{2} \rfloor} a_2^{\lfloor \frac{n}{2} \rfloor} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a_1^{n-2k} a_0^k a_2^k, \end{aligned}$$

where the last equality follows from Pascal's Identity.

Now suppose n is odd. In that case, $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n-2}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$. Equation (2.10) becomes

$$\begin{aligned} P_0^n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-1-k}{k} a_1^{n-2k} a_0^k a_2^k + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-1-k}{k-1} a_1^{n-2k} a_0^k a_2^k \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-k}{k} a_1^{n-2k} a_0^k a_2^k + (-1)^n a_1^n \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-k} \binom{n-k}{k} a_1^{n-2k} a_0^k a_2^k. \end{aligned}$$

□

An equivalent way to write Equation (2.9) is given by Lemma 2.3. To see that Equation (2.11) is equivalent to Equation (2.9), simply let $p = n - 2k$.

Lemma 2.3. *Let P_n be as previously defined. Then,*

$$P_0^n = \sum_{\substack{\forall p,k \geq 0 \\ p+2k=n}} (-1)^{p+k} \binom{p+k}{p} a_0^{n-p-k} a_1^p a_2^k. \tag{2.11}$$

2.1. **Inverting $g_r(n)$ for $r \geq 3$.** We now discuss how to invert Equation (2.1) for arbitrary integers $r \geq 3$. Fortunately, the techniques used for the case of $r = 2$ easily generalize for $r \geq 3$. In particular, Lemma 2.1 becomes Theorem 2.1. Since the proof of Theorem 2.1 is similar to that of Lemma 2.1, we omit the proof. Details are available upon request.

Theorem 2.1. *Let $(f(n))_{n=0}^{\infty}$ be an arbitrary sequence. Let r be a positive integer. Let $(g_r(n))_{n=0}^{\infty}$ be defined by the recurrence $g_r(n) = \sum_{k=0}^{r} a_k f(n-k)$, where $f(n) = 0$ if $n < 0$, and $\{a_i\}_{i=0}^{r}$ are given constants. Define $P_k^n = P_k^n(a_0, a_1, \ldots, a_r)$. Then, for $n \geq 0$,*

$$a_0^{n+1} f(n) = \sum_{k=0}^{n} P_k^n g_r(k), \tag{2.12}$$

where, for $k > 0$,

$$P_k^n = a_0 P_{k-1}^{n-1}, \tag{2.13}$$

and if $k = 0$,

$$P_0^n = -\sum_{i=0}^{r-1} a_{i+1} P_i^{n-1}, \quad n \geq 1, \tag{2.14}$$

with

$$P_0^0 = 1, \qquad P_k^n = 0, \text{ if } k > n. \tag{2.15}$$

We also are able to generalize Lemma 2.3 to the case when $r \geq 3$ and obtain an explicit formula for P_0^n. To arrive at such a generalization, we need to use Pascal's Theorem for multinomial coefficients, namely

$$\begin{aligned}\binom{p_1+p_2+\cdots+p_r}{p_1,p_2,\ldots,p_r} &= \binom{p_1+p_2+\cdots+p_r-1}{p_1-1,p_2,\ldots,p_r} + \binom{p_1+p_2+\cdots+p_r-1}{p_1,p_2-1,\ldots,p_r} \\ &\quad + \cdots + \binom{p_1+p_2+\cdots+p_r-1}{p_1,p_2,\ldots,p_r-1},\end{aligned} \tag{2.16}$$

where the right side of Equation (2.16) is a sum of r terms.

We will also need a generalization of Equation (2.7). This generalization, obtained by the back-substitution technique of Lemma 2.1, is

$$P_k^n = -\sum_{i=1}^{r} a_0^{i-1} a_i P_k^{n-i}, \qquad k \neq n \tag{2.17}$$

$$P_n^n = a_0^n. \tag{2.18}$$

By using Equations (2.16), (2.17), and (2.18) in the appropriate locations of the proof of Lemma 2.2, we are able to prove Theorem 2.2. Details are available upon request.

Theorem 2.2. *Let* $r \geq 1$. *Let* P_0^n *be as defined in Theorem 2.1. Then,*

$$P_0^n = \sum_{\substack{\forall p_i \geq 0, 1 \leq i \leq r \\ \sum_{i=1}^r i p_i = n}} (-1)^{\sum_{i=1}^r p_i} \binom{p_1+p_2+\cdots+p_r}{p_1,p_2,\ldots,p_r} a_0^{n-\sum_{i=1}^r p_i} \prod_{i=1}^r a_i^{p_i}. \tag{2.19}$$

3. Inverting $g(n) = \sum_{k=0}^r f(n-k)$

We will discuss the special case of Equation (2.1) when all $a_i = 1$ for $0 \leq i \leq r$ and r is any positive integer. Gould discussed this situation in [1]. When inverting Equation (0.1), and solving for $f(n)$, Gould discovered that the $g's$ are evaluated at only certain numbers between 0 and n. For a fixed n, Gould found a formula which determines which values of g occur in the inversion. Perhaps the best way to understand Gould's formula is to look at the following example. Assume $r = 2$ and $g(n) = f(n) + f(n-1) + f(n-2)$. Gould showed on page 3 of [1] that

$$f(9) = g(9) - g(8) + g(6) - g(5) + g(3) - g(2) + g(0).$$

Notice that $g(7)$, $g(4)$, and $g(1)$ do not appear. To determine which values of g do appear in $f(9)$, we use the following procedure. Given 9, 8, and 6, the next value to be found is $6 + 8 - 9 = 5$. Then, we evaluate at $5 + 6 - 8 = 3$. Next, we use $3 + 5 - 6 = 2$, and finally $2 + 3 - 5 = 0$. The process terminates at 0 or 1. Gould showed in Theorem 2 of [1] that for any r, the values at which g is evaluated in forming the inverse satisfy a third order recurrence relation of the form $B_n = B_{n-1} + B_{n-2} - B_{n-3}$. Gould proves Theorem 2 via generating functions. We now provide another, slightly simpler, proof of Theorem 2 of

[1]. By using Theorem 2.1 and Equation (2.17), we will be able to see why the inversion of $g_r(n) = \sum_{k=0}^{r} f(n-k)$ is evaluated at the terms provided by $B_n = B_{n-1} + B_{n-2} - B_{n-3}$.

First, when $a_i = 1$ for all i, Equation (2.17) becomes

$$P_k^n = -\sum_{i=1}^{r} P_k^{n-i}. \tag{3.1}$$

We should note that Equation (3.1) implies for $r \geq 2$

$$P_0^0 = 1 \tag{3.2}$$

$$P_0^1 = -1 \tag{3.3}$$

$$P_0^2 = P_0^3 = \ldots = P_0^r = 0. \tag{3.4}$$

Furthermore, by induction on n, Equation (3.1) implies that

$$P_k^n = P_k^{n+r+1}. \tag{3.5}$$

Next, we should note that Equation (1.12) becomes

$$P_k^n = P_{k-1}^{n-1}, \tag{3.6}$$

while Equation (1.11) becomes

$$f(n) = \sum_{k=0}^{n} P_k^n g_r(k). \tag{3.7}$$

Then, by using Equations (3.5) and (3.6) in Equation (3.7), we deduce that

$$P_k^n = P_{k-(r+1)}^n, \qquad k \geq r+1. \tag{3.8}$$

Finally, note that Equations (2.3) to (2.5) and Equation (3.8) are equivalent to Theorem 2 of [1].

4. Inverting $g_r(n) = \sum_{k=0}^{r} a_{r,k} f(n-k)$

We now discuss another generalization of Equation (2.1) Once again, we let $(f(n))_{n=0}^{\infty}$ be an arbitrary sequence and r be a fixed positive integer. We define the sequence $(g_r(n))_{n=0}^{\infty}$ by the recurrence

$$g_r(n) = \sum_{k=0}^{r} a_{r,k} f(n-k), \tag{4.1}$$

where $f(n) = 0$ if $n < 0$, and $\{a_{r,k}\}_{k=0}^{r}$ is the set of coefficients which depend on both r and k. The difference between Equations (2.1) and (4.1) is subtle. In Equation (2.1) the fixed set of coefficients $\{a_k\}_{k=0}^{\infty}$ have no r dependence. Hence, no matter what r we choose, for any $0 \leq i \leq r$, $f(n-i)$ always has the same coefficient, namely a_i. On the other hand, the coefficients in Equation (4.1) do vary with r. For example, take $a_{r,k} = \binom{r}{k}$. Then, Equation

(4.1) implies

$$g_1(n) = \sum_{k=0}^{1} \binom{1}{k} f(n-k) = f(n) + f(n-1)$$
$$g_2(n) = \sum_{k=0}^{2} \binom{2}{k} f(n-k) = f(n) + 2f(n-1) + f(n-2)$$
$$g_3(n) = \sum_{k=0}^{3} \binom{3}{k} f(n-k) = f(n) + 3f(n-1) + 3f(n-2) + ffn-3).$$

Notice that the coefficient of $f(n-1)$ varies depending on r. This variation is the difference between Equation (4.1) and the previous situation.

Our goal is to invert Equation (4.1). Fortunately, the techniques used in Theorems 2.1 and 2.2 directly translate to this situation. In particular, Theorems 2.1 and 2.2 become Theorems 4.1 and 4.2, respectively.

Theorem 4.1. *Let $(f(n))_{n=0}^{\infty}$ be an arbitrary sequence. Let r be a positive integer. Let $(g_r(n))_{n=0}^{\infty}$ be defined by the recurrence $g_r(n) = \sum_{k=0}^{r} a_{r,k} f(n-k)$, where $f(n) = 0$ if $n < 0$, and $\{a_{r,k}\}_{k=0}^{r}$ is an array of given constants. Then, for $n \geq 0$,*

$$a_{r,0}^{n+1} f(n) = \sum_{k=0}^{n} P_k^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) g_r(k), \tag{4.2}$$

where, for $k > 0$,

$$P_k^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) = a_{r,0} P_{k-1}^{n-1}(a_{r,0}, a_{r,1}, \ldots, a_{r,r}), \tag{4.3}$$

and if $k = 0$,

$$P_0^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) = -\sum_{i=0}^{r-1} a_{r,r+1} P_i^{n-1}(a_{r,0}, a_{r,1}, \ldots, a_{r,r}), \quad n \geq 1, \tag{4.4}$$

with

$$P_0^0(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) = 1, \qquad P_k^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) = 0, \text{ if } k > n. \tag{4.5}$$

Theorem 4.2. *Let $r \geq 1$. Let $P_0^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r})$ be as defined in Theorem 4.1. Then,*

$$P_0^n(a_{r,0}, a_{r,1}, \ldots, a_{r,r}) = \sum_{\substack{\forall p_i \geq 0, 1 \leq i \leq r \\ \sum_{i=1}^{r} i p_i = n}} (-1)^{\sum_{i=1}^{r} p_i} \binom{p_1 + p_2 + \cdots + p_r}{p_1, p_2, \ldots, p_r} a_{r,0}^{n - \sum_{i=1}^{r} p_i} \prod_{i=1}^{r} a_{r,i}^{p_i}. \tag{4.6}$$

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