BEATTY SEQUENCES, FIBONACCI NUMBERS, AND THE GOLDEN RATIO

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ABSTRACT. \((\lfloor n\phi \rfloor)_{n \geq 1}\) and \((\lfloor n\phi^2 \rfloor)_{n \geq 1}\) are well-known complementary Beatty sequences. An infinite set of complementary Beatty sequences, based on a generalization of ratios of Fibonacci numbers and higher powers of \(\phi\), is proved. An open problem posed by Clark Kimberling, the Swappage Problem, is resolved in the affirmative as a special case of this set of complementary Beatty sequences.

1. Introduction

A Beatty sequence \(\llbracket 9, 1, 10 \rrbracket\) is generated by an irrational \(\alpha > 1\) as follows:
\[
(\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots) = (\lfloor n\alpha \rfloor)_{n \geq 1}.
\]
Letting \(\beta\) be the number satisfying \(1/\alpha + 1/\beta = 1\), the sequences \((\lfloor n\alpha \rfloor)_{n \geq 1}\) and \((\lfloor n\beta \rfloor)_{n \geq 1}\) are complementary Beatty sequences. See for example [6].

It is well-known that if \(\alpha = \phi\), where \(\phi\) denotes the golden ratio, then \(\beta = \alpha + 1\). The corresponding sequence \((\lfloor n\alpha \rfloor)_{n \geq 1} = (1, 3, 4, 6, \ldots)\), is the lower Wythoff sequence [7]. The complementary Beatty sequence, \((\lfloor n\beta \rfloor)_{n \geq 1} = (2, 5, 7, 10, \ldots)\), is the upper Wythoff sequence [7].

2. Beatty Sequences

As noted above, when \(\alpha = \phi\), we obtain \(\beta = \alpha + 1\). From the identity \(\phi^2 = \phi + 1\), we may rewrite \(\beta\) as \(\beta = \phi^2\). These complementary Beatty sequences form the basis for Theorem 2.2, which uses the following well-known result.

Lemma 2.1.
\[
\phi^{k+1} = F_{k+1}\phi + F_k.
\]

Proof. This is easily proved by induction. For the base case, observe that \(F_1 = F_2 = 1\) and \(\phi^2 = \phi + 1\). For the inductive case, assume equality holds for all \(k < n\). Then,
\[
\phi^{n+1} = \phi(F_n\phi + F_{n-1}) \\
= F_n(\phi + 1) + F_{n-1}\phi \\
= (F_n + F_{n-1})\phi + F_n \\
= F_{n+1}\phi + F_n.
\]
\(\square\)
Theorem 2.2. For all $i \geq 1$,

$$\left(\left\lfloor \frac{n\phi^i}{F_{i+1}} \right\rfloor\right)_{n \geq 1} \quad \text{and} \quad \left(\left\lfloor \frac{n\phi^{i+1}}{F_i} \right\rfloor\right)_{n \geq 1}$$

are complementary Beatty sequences.

Proof.

$$\frac{F_{k+1}}{\phi^k} + \frac{1}{\beta} = 1$$

$$F_{k+1}\beta + \phi^k = \phi^k\beta$$

$$\beta = \frac{\phi^k}{\phi^k - F_{k+1}}$$

$$= \frac{\phi^{k+1} - F_{k+1}\phi}{\phi^{k+1}}$$

$$= \frac{\phi^{k+1} - \phi^{k+1} + F_k}{\phi}$$

by Lemma 2.1

$$\beta = \frac{\phi^{k+1}}{F_k}.$$  

□

This theorem can be applied as a special case to an open problem posed by Clark Kimberling, the Swappage Problem [8], which is resolved in Section 3. Through some mathematical manipulation, an alternative derivation of the swappage sequence is provided which is used to resolve the Swappage Problem in the affirmative.

3. The Swappage Problem

Problem Statement. Let $L = (1, 3, 4, 6, 8, \ldots)$ be the Lower Wythoff Sequence [2]. Similarly, let $U$ be the complement $L'$ of $L$; i.e., $U = (2, 5, 7, 10, \ldots)$ is the Upper Wythoff Sequence [3]. For each odd $U(n)$, let $L(m)$ be the least number in $L$ such that after swapping $U(n)$ and $L(m)$, the resulting new sequences are both increasing. The resulting sequence derived by swapping these elements, called the swappage of $L$, is $V = (2, 4, 6, 10, 12, 14, 18, 20, 22, 26, \ldots)$ [5].

Let $S(n) = \frac{V(n)}{2}$ for every $n$. Is the complement of $S$ (in the set of nonnegative integers) the same set of numbers that comprise the sequence

$$\left(\left\lfloor n\phi^3 \right\rfloor\right)_{n \geq 0} = (0, 4, 8, 12, 16, 21, 25, 29, \ldots)?$$

Solution. The sequence $V$ is generated by the swapping algorithm described in the problem statement. We first prove that $V$ can be derived by an entirely different method, one that requires no swapping. We then prove that $S' = \left(\left\lfloor n\phi^3 \right\rfloor\right)_{n \geq 0}$. The following sequence, labeled $W$, is also derived from $U$ without any swapping.

$$W(n) = \begin{cases} U(n) & \text{if } U(n) \text{ is even}, \\ U(n) - 1 & \text{if } U(n) \text{ is odd}. \end{cases}$$

The first few terms of $W$ are given below:
BEATTY SEQUENCES, FIBONACCI NUMBERS, AND THE GOLDEN RATIO

\[ U = (2, 5, 7, 10, 13, 15, 18, 20, \ldots) \]
\[ W = (2, 4, 6, 10, 12, 14, 18, 20, \ldots). \]

We shall now prove that \( W = V. \)

Proof. First, note that if \( U(n) \) is even, then \( U(n) = V(n) = W(n). \) So, assume that \( U(n) \) is odd. Observe that for all \( n > 0, U(n) - 1 \in L, \) because \( U(n) = \lfloor n\phi^2 \rfloor \) and \( \phi^2 > 2.6. \) Hence, \( U(n) \in U \implies U(n) - 1 \notin U. \) \( U(n) - 1 \in L \) is trivial and follows directly from the problem statement where \( U \) is defined as the complement of \( L. \)

For the odd \( U(n), \) we prove equality by induction. For the base case, observe that \( U(2) = 5 \) and \( V(2) = W(2) = 4 \) as shown in the above sequences. Assume that \( V(k) = W(k) \) for \( k < n \) and consider the case where \( U(n) \) is odd. According to the definition of \( W, \) \( W(n) = L(m) \) for some element \( m \) such that \( L(m) = U(n) - 1. \) Since \( L(m) = U(n) - 1, \) if \( U(n) \) and \( L(m) \) are swapped, the resulting sequences, \( U_n \) and \( L_n, \) are both increasing sequences. For instance:

\[ L_n = (1, 3, 5, 7, \ldots, U(n), \ldots) \]
\[ U_n = (2, 4, 6, \ldots, U(n) - 1, \ldots). \]

Now consider which element is swapped to generate \( V(n). \) In the Problem Statement, we generate \( V \) by swapping the least element of \( L \) such that \( L \) and \( U \) both remain increasing sequences. Note that \( L(m) \in L \) when we are choosing an element to swap for \( V(n), \) because for all \( k < n, W(k) = V(k) \) and \( W(k) \neq L(m). \) Since \( L(m) \) can be swapped with \( U(n) \) while maintaining increasing sequences, we need to consider swapping \( L(k) \) with \( U(n) \) only for \( k < m. \) However, any such \( L(k) \) would result in the sequence \( L = (\ldots, U(n), \ldots, L(m), \ldots) \) and since \( L(m) = U(n) - 1, \) then \( L \) is no longer an increasing sequence. Therefore, we must swap \( L(m) \) and \( U(n). \) Thus, the sequence \( V \) has the same elements as sequence \( W. \) \( \square \)

Let us now obtain an alternative expression for \( S. \) Observe that because of the relationship between \( U \) and \( W, \) \( W = \left( 2 \left\lfloor \frac{U(n)}{2} \right\rfloor \right)_{n \geq 1} = \left( 2 \left\lceil \frac{n\phi^2}{2} \right\rceil \right)_{n \geq 1}. \) Because \( W = V, \) \( S(n) = \frac{W(n)}{2}, \) so

\[ S = \left( \left\lfloor \frac{n\phi^2}{2} \right\rfloor \right)_{n \geq 1}. \]

Hence, \( S \) and \( S' \) are complementary Beatty sequences as a special case of Theorem 2.2, which proves the problem statement. Specifically,

\[ \left( \left\lfloor \frac{n\phi^3}{F_3} \right\rfloor \right)_{n \geq 1} \text{ and } \left( \left\lfloor \frac{n\phi^3}{F_2} \right\rfloor \right)_{n \geq 1} \]

are complementary sequences. Note that since \( 0 \notin S, \) \( S' = \left( \left\lfloor n\phi^3 \right\rfloor \right)_{n \geq 0} \text{ if } S' \) is defined for the set of nonnegative integers as in the problem statement and integer sequence A004976 [4].

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References

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