SOME NEW REMARKS ABOUT THE DYING RABBIT PROBLEM

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ABSTRACT. In this paper, we studied a generalization of the Fibonacci sequence in which rabbits become mature h months after their birth and die k months after they have matured. We give a general recurrence relation for these sequences. By using a companion matrix and generating matrix, we derive two classes of identities for the generalized Fibonacci numbers.

1. Introduction

Fibonacci numbers arose as the answer to a problem proposed by Leonardo of Pisa (also known as Fibonacci) who asked for the number of rabbits at the nth month if there is one pair of rabbits at the 0th month which become mature one month later and that breeds another pair in each of the succeeding months, and if these new pairs breed again in the second month following birth. It is well-known that the number of pairs of rabbits at the nth month is given by F_n, with F_n satisfying the recurrence relation:

$$\begin{aligned} F_0 &= 0,\ F_1 = 1; \\ F_n &= F_{n-1} + F_{n-2}, \text{ for every } n \geq 2. \end{aligned} \tag{1.1}$$

If the roots of the characteristic equation $\lambda^2 = \lambda + 1$ are $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$, then the Binet's formula is

$$F_n = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.2}$$

Bicknell-Johnson and Spears [2] used elementary matrix operations to make simple derivations of entries of identities for the generalized Fibonacci numbers $G_n = G_{n-1} + G_{n\text{-}c}$. Kalman [5] derived a number of closed-form formulas for the generalized sequence by the companion matrix method.

Hoggatt and Lind [4], Alfred [1], and Cohn [3] considered the so-called "dying rabbit problem". Recently, Oller-Marcen [7] studied a generalization of the Fibonacci sequence in which rabbits become mature h months after their birth and die k months after they have matured, a general recurrence relation for these sequences was given. But we discover that the conclusions in [7] are incorrect. In this paper, we will review the "dying rabbit problem", and correct the drawbacks in the paper [7]. The recurrence relation is given in Section 2, identities on the generalized Fibonacci numbers are given in Section 3, and we conclude the paper in the last section.

2. The Recurrence Relation

Following the notations in [7], assuming h, $k \geq 2$, let $C_n^{(k,h)}$ be the number of pairs of rabbits at the nth month. Obviously we have $C_0^{(k,h)} = C_1^{(k,h)} = \cdots = C_{h-1}^{(k,h)} = 1$. $C_{n-h}^{(k,h)}$ is the number of pairs of rabbits at the $(n-h)$th month. From the point of view of the nth month, $C_{n-h}^{(k,h)}$ also denotes the number of pairs of rabbits which just came into the world (newly added rabbits), and $C_{n-h-k}^{(k,h)}$ denotes the numbers of pairs of the rabbits which just died. $C_n^{(k,h)}$ is correlated to the numbers of the preceding month $C_{n-1}^{(k,h)}$, newly added rabbits $C_{n-h}^{(k,h)}$ and died rabbits $C_{n-h-k}^{(k,h)}$. There is the following relation.

Theorem 2.1.

$$C_n^{(k,h)} = \begin{cases} 1, & \text{if } 0 \le n \le h-1; \\ C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)}, & \text{if } h \le n \le h+k-1; \\ C_{n-1}^{(k,h)} + C_{n-h-1}^{(k,h)} - C_0^{(k,h)}, & \text{if } n = h+k; \\ C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)} - C_{n-h-k}^{(k,h)}, & \text{if } n \ge h+k+1. \end{cases} \tag{2.1}$$

Proof. If $0 \le n \le h-1$, the result is clear since the only pair of rabbits is the initial one.

If $h \le n \le h+k-1$, no rabbits have died yet, so the number of pairs at the nth month is the sum of the pairs at the preceding month $C_{n-1}^{(k,h)}$, and those bred by the pairs which are mature at this point, i.e. $C_{n-h}^{(k,h)}$. Thus, $C_n^{(k,h)} = C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)}$.

If $n = h+k$, the initial pair of rabbits dies, so they do breed, the number of rabbits is $C_{n-1}^{(k,h)} + C_{n-h-1}^{(k,h)} - C_0^{(k,h)}$.

If $n \ge h+k+1$, the number of rabbits at the nth month can be computed as the sum of the pairs at the preceding month $C_{n-1}^{(k,h)}$, and those bred by the pairs which are mature at this point, i.e. $C_{n-h}^{(k,h)}$. (It is the number of pairs of rabbits which just came into the world.) Finally, the number of pairs of rabbits which died at this point, i.e. $C_{n-h-k}^{(k,h)}$, must be subtracted. Thus, $C_n^{(k,h)} = C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)} - C_{n-h-k}^{(k,h)}$. □

Example 2.2. *If $h = 2$ and $k = 3$, the recurrence relation is*

$$C_n^{(k,h)} = \begin{cases} 1, & \text{if } 0 \le n \le 1; \\ C_{n-1}^{(3,2)} + C_{n-2}^{(3,2)}, & \text{if } 2 \le n \le 4; \\ C_{n-1}^{(3,2)} + C_{n-3}^{(3,2)} - C_0^{(3,2)}, & \text{if } n = 5; \\ C_{n-1}^{(3,2)} + C_{n-2}^{(3,2)} - C_{n-5}^{(3,2)}, & \text{if } n \ge 6. \end{cases} \tag{2.2}$$

The beginning terms of $C_n^{(3,2)}$ are: $1, 1, 2, 3, 5, 6, 10, 14, 21, 30, 45, \ldots$.

Example 2.3. *If $h = 4$ and $k = 7$, the recurrence relation is*

$$C_n^{(k,h)} = \begin{cases} 1, & \text{if } 0 \le n \le 3; \\ C_{n-1}^{(7,4)} + C_{n-4}^{(7,4)}, & \text{if } 2 \le n \le 10; \\ C_{n-1}^{(7,4)} + C_{n-5}^{(7,4)} - C_0^{(7,4)}, & \text{if } n = 11; \\ C_{n-1}^{(7,4)} + C_{n-4}^{(7,4)} - C_{n-11}^{(7,4)}, & \text{if } n \ge 12. \end{cases} \tag{2.3}$$

The beginning terms of $C_n^{(7,4)}$ are: $1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 17, 23, 32, 45, 60, 80, 108, \ldots$.

3. Identities on the Generalized Fibonacci Numbers

In this section, we consider the formula (2.1) as two cases: one case is for $0 \le n \le h+k-1$, the other case is for $n \ge h+k+1$.

Case 1. If $0 \le n \le h+k-1$, there are the following identities.

Proposition 3.1. *If $h \leq k-1$, then*

$$C_{2h}^{(k,h)} = C_h^{(k,h)} + h + 1. \tag{3.1}$$

Proof. The condition $h \leq k-1$ is equal to $2h \leq h+k-1$; this is the case in (2.1), that is $C_n^{(k,h)} = C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)}$. The initial conditions are $C_0^{(k,h)} = C_1^{(k,h)} = \cdots = C_{h-1}^{(k,h)} = 1$. Thus, there are $C_h^{(k,h)} = C_{h-1}^{(k,h)} + C_0^{(k,h)} = 2, C_{h+1}^{(k,h)} = C_h^{(k,h)} + C_1^{(k,h)} = 3, \cdots, C_{2h}^{(k,h)} = C_{2h-1}^{(k,h)} + C_h^{(k,h)} = h+3$. Summing all these $2h$ equalities, one obtains identity (3.1). □

Proposition 3.2. *If $mh \leq k-1$, then*

$$C_{(m+1)h}^{(k,h)} - 1 = C_0^{(k,h)} + C_1^{(k,h)} + C_2^{(k,h)} + \cdots + C_{mh}^{(k,h)}. \tag{3.2}$$

Proof. The condition $mh \leq k-2$ is equal to $(m+1)h \leq h+k-1$. This is the case in (2.1), that is $C_n^{(k,h)} = C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)}$, the identity (3.2) is the identity (3.2) in [5]. □

Case 2. If $n \geq h+k+1$, we can derive two classes of identities for the generalized Fibonacci numbers (2.1) by using a companion matrix [5] and generating matrix [6], respectively.

The $(h+k)\times(h+k)$ companion matrix of the characteristic equation $\lambda^{h+k} = \lambda^{h+k-1} + \lambda^k - 1$, for the recurrence relation $C_n^{(k,h)} = C_{n-1}^{(k,h)} + C_{n-h}^{(k,h)} - C_{n-h-k}^{(k,h)}$ is,

$$A = \begin{bmatrix} 0 & 1 & & & & & & & & \\ & 0 & 1 & & & & & & & \\ & & 0 & \ddots & & & & & & \\ & & & 0 & \ddots & & & & & \\ & & & & \ddots & \ddots & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & & 0 & 1 \\ -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & & 0 & 1 \end{bmatrix} \tag{3.3}$$

and its augmented matrix is

$$A_n^* = \begin{bmatrix} C_n^{(k,h)} & C_{n+1}^{(k,h)} & \cdots & C_{n+h+k-1}^{(k,h)} & C_{n+w}^{(k,h)} \\ C_{n+1}^{(k,h)} & C_{n+2}^{(k,h)} & \cdots & C_{n+h+k}^{(k,h)} & C_{n+w+1}^{(k,h)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n+h+k-1}^{(k,h)} & C_{n+h+k}^{(k,h)} & \cdots & C_{n+2(h+k-1)}^{(k,h)} & C_{n+w+h+k-2}^{(k,h)} \end{bmatrix}. \tag{3.4}$$

We can get that $|A| = \pm 1$, $A_{n+1}^* = AA_n^*$, and $A_n^* = AA_{n-1}^* = A^2 A_{n-2}^* = \cdots = A^n A_0^*$.

Suppose we want another identity. Let w be an arbitrary given positive integer.

Class 1.

$$C_{n+w}^{(k,h)} = \alpha_1 C_n^{(k,h)} + \alpha_2 C_{n-1}^{(k,h)} + \cdots + \alpha_{h+k-1} C_{n-h-k+2}^{(k,h)} + \alpha_{h+k} C_{n-h-k+1}^{(k,h)}, \quad \text{for } n \geq h+k+1. \tag{3.5}$$

In order to solve for the constants $\alpha_1, \alpha_2, \ldots \alpha_{h+k}$, we use elementary row operations on the augmented matrix A_0^*,

$$A_0^* = \begin{bmatrix} C_0^{(k,h)} & C_1^{(k,h)} & \cdots & C_{h+k-1}^{(k,h)} & C_{w+h+k-1}^{(k,h)} \\ C_1^{(k,h)} & C_2^{(k,h)} & \cdots & C_{h+k}^{(k,h)} & C_{w+h+k}^{(k,h)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{h+k-1}^{(k,h)} & C_{h+k}^{(k,h)} & \cdots & C_{2(h+k-1)}^{(k,h)} & C_{w+2(h+k-1)}^{(k,h)} \end{bmatrix}, \tag{3.6}$$

to obtain a class of identities (for various w values).

Example 3.3. *If $h = 2$ and $k = 3$, the identities (3.5) are*

$$C_{n+w}^{(3,2)} = \alpha_1 C_n^{(3,2)} + \alpha_2 C_{n-1}^{(3,2)} + \alpha_3 C_{n-2}^{(3,2)} + \alpha_4 C_{n-3}^{(3,2)} + \alpha_5 C_{n-4}^{(3,2)}, \tag{3.7}$$

with $n \geq 6$, and the augmented matrix A_0^*,

$$A_0^* = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & C_{w+4}^{(3,2)} \\ 1 & 2 & 3 & 5 & 6 & C_{w+5}^{(3,2)} \\ 2 & 3 & 5 & 6 & 10 & C_{w+6}^{(3,2)} \\ 3 & 5 & 6 & 10 & 14 & C_{w+7}^{(3,2)} \\ 5 & 6 & 10 & 14 & 21 & C_{w+8}^{(3,2)} \end{bmatrix}. \tag{3.8}$$

For different positive integers w, we get different identity classes (with $n \geq 6$):

$$\begin{aligned} C_{n+6}^{(3,2)} &= 4C_n^{(3,2)} + 5C_{n-1}^{(3,2)} + 4C_{n-2}^{(3,2)} + 2C_{n-3}^{(3,2)}; \\ C_{n+7}^{(3,2)} &= 5C_n^{(3,2)} + 8C_{n-1}^{(3,2)} + 6C_{n-2}^{(3,2)} + 4C_{n-3}^{(3,2)}; \\ C_{n+8}^{(3,2)} &= 8C_n^{(3,2)} + 11C_{n-1}^{(3,2)} + 9C_{n-2}^{(3,2)} + 5C_{n-3}^{(3,2)}; \\ C_{n+9}^{(3,2)} &= 11C_n^{(3,2)} + 17C_{n-1}^{(3,2)} + 13C_{n-2}^{(3,2)} + 8C_{n-3}^{(3,2)}; \\ C_{n+10}^{(3,2)} &= 17C_n^{(3,2)} + 24C_{n-1}^{(3,2)} + 19C_{n-2}^{(3,2)} + 11C_{n-3}^{(3,2)}; \\ &\cdots. \end{aligned}$$

Example 3.4. *If $h = 4$ and $k = 4$, the recurrence relation is*

$$C_n^{(k,h)} = \begin{cases} 1, & \text{if } 0 \leq n \leq 3; \\ C_{n-1}^{(4,4)} + C_{n-4}^{(4,4)}, & \text{if } 4 \leq n \leq 7; \\ C_{n-1}^{(4,4)} + C_{n-5}^{(4,4)} - C_0^{(4,4)}, & \text{if } n = 8; \\ C_{n-1}^{(4,4)} + C_{n-4}^{(4,4)} - C_{n-8}^{(4,4)}, & \text{if } n \geq 9. \end{cases} \tag{3.9}$$

The beginning terms of $C_n^{(4,4)}$ are: $1, 1, 1, 1, 2, 3, 4, 5, 5, 7, 10, 14, 17, 21, 27, 36, 48, 62, 79, \ldots$. *The identities (3.5) are*

$$C_{n+w}^{(4,4)} = \alpha_1 C_n^{(4,4)} + \alpha_2 C_{n-1}^{(4,4)} + \cdots + \alpha_8 C_{n-7}^{(4,4)}, n \geq 9. \tag{3.10}$$

By using elementary row operations on the augmented matrix A_0^*,

$$A_0^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & C_{w+7}^{(4,4)} \\ 1 & 1 & 1 & 2 & 3 & 4 & 5 & 5 & C_{w+8}^{(4,4)} \\ 1 & 1 & 2 & 3 & 4 & 5 & 5 & 7 & C_{w+9}^{(4,4)} \\ 1 & 2 & 3 & 4 & 5 & 5 & 7 & 10 & C_{w+10}^{(4,4)} \\ 2 & 3 & 4 & 5 & 5 & 7 & 10 & 14 & C_{w+11}^{(4,4)} \\ 3 & 4 & 5 & 5 & 7 & 10 & 14 & 17 & C_{w+12}^{(4,4)} \\ 4 & 5 & 5 & 7 & 10 & 14 & 17 & 21 & C_{w+13}^{(4,4)} \\ 5 & 5 & 7 & 10 & 14 & 17 & 21 & 27 & C_{w+14}^{(4,4)} \end{bmatrix}. \tag{3.11}$$

For different positive integers w, we get different identity classes (with $n \geq 9$):

$C_{n+12}^{(4,4)} = 4C_n^{(4,4)} + 5C_{n-1}^{(4,4)} + 7C_{n-2}^{(4,4)} + 10C_{n-3}^{(4,4)} + 9C_{n-4}^{(4,4)} + 7C_{n-5}^{(4,4)} + 4C_{n-6}^{(4,4)}$;
$C_{n+13}^{(4,4)} = 5C_n^{(4,4)} + 7C_{n-1}^{(4,4)} + 10C_{n-2}^{(4,4)} + 13C_{n-3}^{(4,4)} + 11C_{n-4}^{(4,4)} + 8C_{n-5}^{(4,4)} + 4C_{n-6}^{(4,4)}$;
$C_{n+14}^{(4,4)} = 7C_n^{(4,4)} + 10C_{n-1}^{(4,4)} + 13C_{n-2}^{(4,4)} + 16C_{n-3}^{(4,4)} + 13C_{n-4}^{(4,4)} + 9C_{n-5}^{(4,4)} + 5C_{n-6}^{(4,4)}$;
$C_{n+15}^{(4,4)} = 10C_n^{(4,4)} + 13C_{n-1}^{(4,4)} + 16C_{n-2}^{(4,4)} + 20C_{n-3}^{(4,4)} + 16C_{n-4}^{(4,4)} + 12C_{n-5}^{(4,4)} + 7C_{n-6}^{(4,4)}$;
$C_{n+16}^{(4,4)} = 13C_n^{(4,4)} + 16C_{n-1}^{(4,4)} + 20C_{n-2}^{(4,4)} + 26C_{n-3}^{(4,4)} + 22C_{n-4}^{(4,4)} + 17C_{n-5}^{(4,4)} + 10C_{n-6}^{(4,4)}$;
$\cdots$.

Using the $(h+k) \times (h+k)$ generating matrix

$$B = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ x_{h+k} & x_{h+k-1} & \cdots & x_2 & x_1 \end{bmatrix} \tag{3.12}$$

and the augmented matrix B_n^*

$$B_n^* = \begin{bmatrix} C_{wn}^{(k,h)} & C_{w(n+1)}^{(k,h)} & \cdots & C_{w(n+h+k-1)}^{(k,h)} & C_{w(n+h+k)}^{(k,h)} \\ C_{w(n+1)}^{(k,h)} & C_{w(n+2)}^{(k,h)} & \cdots & C_{w(n+h+k)}^{(k,h)} & C_{w(n+h+k+1)}^{(k,h)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{w(n+h+k-1)}^{(k,h)} & C_{w(n+h+k)}^{(k,h)} & \cdots & C_{w(n+2(h+k-1))}^{(k,h)} & C_{w(n+2(h+k)-1))}^{(k,h)} \end{bmatrix}. \tag{3.13}$$

Suppose we want another identity. Let w be an arbitrary given positive integer.

Class 2.

$$C_{wn}^{(k,h)} = x_1 C_{w(n-1)}^{(k,h)} + x_2 C_{w(n-2)}^{(k,h)} + \cdots + x_{h+k} C_{w(n-(h+k))}^{(k,h)}, \text{ for } n \geq h+k+1. \tag{3.14}$$

We can obtain an analogous result. $|B| = \pm x_r$ (suppose $x_r \neq 0$), $B_{w(n+1)}^* = BB_{wn}^*$, and $B_{wn}^* = BB_{w(n-1)}^* = B^2 B_{w(n-2)}^* = \cdots = B^n B_0^*$. In order to solve for the constants $x_1, x_2, \cdots x_{h+k}$, we use elementary row operations on the augmented matrix B_0^*,

$$B_0^* = \begin{bmatrix} C_0^{(k,h)} & C_w^{(k,h)} & \cdots & C_{w(h+k-2)}^{(k,h)} & C_{w(h+k-1)}^{(k,h)} \\ C_w^{(k,h)} & C_{w2}^{(k,h)} & \cdots & C_{w(h+k-1)}^{(k,h)} & C_{w(h+k)}^{(k,h)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{w(h+k-2)}^{(k,h)} & C_{w(h+k-1)}^{(k,h)} & \cdots & C_{w2(h+k-2)}^{(k,h)} & C_{w(2(h+k)-3)}^{(k,h)} \end{bmatrix}. \tag{3.15}$$

Example 3.5. *If $h = 2$ and $k = 3$, the identities* (3.14) *are*

$$C_{wn}^{(3,2)} = x_1 C_{w(n-1)}^{(3,2)} + x_2 C_{w(n-2)}^{(3,2)} + x_3 C_{w(n-3)}^{(3,2)} + x_4 C_{w(n-4)}^{(3,2)} + x_5 C_{w(n-5)}^{(3,2)}, \text{ for } n \geq 6. \quad (3.16)$$

By using elementary row operations on the augmented matrix B_0^*, for different positive integers w, we get different identity classes (with $n \geq 6$):

$$C_{2n}^{(3,2)} = 2C_{2(n-1)}^{(3,2)} + C_{2(n-2)}^{(3,2)} - C_{2(n-3)}^{(3,2)} - C_{2(n-4)}^{(3,2)};$$
$$C_{3n}^{(3,2)} = 3C_{3(n-1)}^{(3,2)} + C_{3(n-2)}^{(3,2)} - 2C_{3(n-3)}^{(3,2)} + C_{3(n-4)}^{(3,2)};$$
$$C_{4n}^{(3,2)} = 6C_{4(n-1)}^{(3,2)} - 7C_{4(n-2)}^{(3,2)} + 3C_{4(n-3)}^{(3,2)} - C_{4(n-4)}^{(3,2)};$$
$$C_{5n}^{(3,2)} = 5C_{5(n-1)}^{(3,2)} + 11C_{5(n-2)}^{(3,2)} + 6C_{5(n-3)}^{(3,2)} + C_{5(n-4)}^{(3,2)};$$
$$\cdots.$$

4. Conclusion

In this paper, we studied a generalization of the Fibonacci sequence in which rabbits become mature h months after their birth and die k months after they have matured. We give a general recurrence relation for these sequences. By using a companion matrix and generating matrix, we derive two classes of identities for the generalized Fibonacci numbers.

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