APPROXIMATING EULER'S CONSTANT

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Abstract. We use the elementary technique of telescoping series and the Maclaurin series for $\log(1+x)$ to obtain sharp estimates for Euler's constant.

1. Introduction

Many papers (see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) have appeared in recent years giving approximations to Euler's constant. The aim of this note is to show how one can use the elementary technique of summing telescoping series to obtain inequalities sharper than those obtained in any of the above-mentioned articles.

We shall prove the following three inequalities.

For $n \geq 2$,

$$\frac{1}{2n+\frac{1}{3}+\frac{1}{18n}} < \sum_{k=1}^{n} \frac{1}{k} - \log n - \gamma < \frac{1}{2n+\frac{1}{3}+\frac{1}{32n}}, \tag{1.1}$$

$$\frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{6n(n+1)}} < \sum_{k=1}^{n} \frac{1}{k} - \log \sqrt{n(n+1)} - \gamma < \frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{4n(n+1)}} \tag{1.2}$$

and

$$\frac{1}{24n(n+1)+\frac{51}{5}-\frac{1}{n(n+1)}} < \sum_{k=1}^{n} \frac{1}{k} - \log\left(n+\frac{1}{2}\right) - \gamma < \frac{1}{24n(n+1)+\frac{51}{5}-\frac{3}{2n(n+1)}}. \tag{1.3}$$

In deriving these results, we use nothing more than the Maclaurin series for $\log(1+x)$.

Furthermore, the inequality (1.3) enables us to prove results such as

$$\left(n+\frac{1}{2}\right)^2 \left(\sum_{k=1}^{n} \frac{1}{k} - \log\left(n+\frac{1}{2}\right) - \gamma\right) \quad \text{is an increasing sequence}$$

and

$$\left(\left(n+\frac{1}{2}\right)^2 + \frac{1}{2}\right) \left(\sum_{k=1}^{n} \frac{1}{k} - \log\left(n+\frac{1}{2}\right) - \gamma\right) \quad \text{is a decreasing sequence.}$$

The first of these is stated in [4], the second is an improvement on a result conjectured there. (Both sequences have $\frac{1}{24}$ as limit.)

2. Proofs of the Inequalities

In what follows, an inequality written

$$a >^* b$$

can be verified by computing the rational function $a-b$ and observing that $a-b>0$. Similarly, an inequality written

$$a <^* b$$

can be verified by computing the rational function $b-a$ and observing that $b-a>0$.

Let

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

Then

$$\begin{aligned}\gamma_n - \gamma_{n+1} &= \log\left(1+\frac{1}{n}\right) - \frac{1}{n+1}\\ &= -\log\left(1-\frac{1}{n+1}\right) - \frac{1}{n+1}\\ &= \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \cdots .\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n - \gamma_{n+1} &> \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \frac{1}{5(n+1)^5} + \frac{1}{6(n+1)^6}\\ &>^* \frac{1}{2n+\frac{1}{3}+\frac{1}{18n}} - \frac{1}{2(n+1)+\frac{1}{3}+\frac{1}{18(n+1)}}\end{aligned}$$

and

$$\begin{aligned}\gamma_n - \gamma_{n+1} &< \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \frac{1}{5(n+1)^5} \cdot \frac{1}{1-\frac{1}{n+1}}\\ &= \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \frac{1}{5n(n+1)^4}\\ &<^* \frac{1}{2n+\frac{1}{3}+\frac{1}{32n}} - \frac{1}{2(n+1)+\frac{1}{3}+\frac{1}{32(n+1)}}\end{aligned}$$

for $n \geq 2$.

That is, for $n \geq 2$,

$$\frac{1}{2n+\frac{1}{3}+\frac{1}{18n}} - \frac{1}{2(n+1)+\frac{1}{3}+\frac{1}{18(n+1)}} < \gamma_n - \gamma_{n+1} < \frac{1}{2n+\frac{1}{3}+\frac{1}{32n}} - \frac{1}{2(n+1)+\frac{1}{3}+\frac{1}{32(n+1)}}.$$

If we sum this from n to ∞, we obtain

$$\frac{1}{2n+\frac{1}{3}+\frac{1}{18n}} < \gamma_n - \gamma < \frac{1}{2n+\frac{1}{3}+\frac{1}{32n}}$$

for $n \geq 2$, which is (1.1).

Now let

$$\gamma_n^* = \sum_{k=1}^{n} \frac{1}{k} - \log\sqrt{n(n+1)}.$$

Then

$$\begin{aligned}\gamma_n^* - \gamma_{n+1}^* &= \frac{1}{2}\log(n+2) - \frac{1}{2}\log n - \frac{1}{n+1}\\ &= \frac{1}{2}\log\left(\frac{1+\frac{1}{n+1}}{1-\frac{1}{n+1}}\right) - \frac{1}{n+1}\\ &= \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \frac{1}{7(n+1)^7} + \cdots .\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n^* - \gamma_{n+1}^* &> \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \frac{1}{7(n+1)^7} + \frac{1}{9(n+1)^9}\\ &>^* \frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{6n(n+1)}} - \frac{1}{6(n+1)(n+2)+\frac{6}{5}-\frac{1}{6(n+1)(n+2)}}\end{aligned}$$

for $n \geq 2$, and

$$\begin{aligned}\gamma_n^* - \gamma_{n+1}^* &< \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \frac{1}{7(n+1)^7} \cdot \frac{1}{1-\frac{1}{(n+1)^2}}\\ &= \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \frac{1}{7n(n+1)^5(n+2)}\\ &<^* \frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{4n(n+1)}} - \frac{1}{6(n+1)(n+2)+\frac{6}{5}-\frac{1}{4(n+1)(n+2)}}.\end{aligned}$$

That is, for $n \geq 2$,

$$\frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{6n(n+1)}}-\frac{1}{6(n+1)(n+2)+\frac{6}{5}-\frac{1}{6(n+1)(n+2)}}$$
$$<\gamma_n^*-\gamma_{n+1}^*$$
$$<\frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{4n(n+1)}}-\frac{1}{6(n+1)(n+2)+\frac{6}{5}-\frac{1}{4(n+1)(n+2)}}.$$

If we sum this from n to ∞, we obtain

$$\frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{6n(n+1)}}<\gamma_n^*-\gamma<\frac{1}{6n(n+1)+\frac{6}{5}-\frac{1}{4n(n+1)}},$$

for $n \geq 2$, which is (1.2).

Finally, let

$$\gamma_n^{**}=\sum_{k=1}^{n}\frac{1}{k}-\log\left(n+\frac{1}{2}\right).$$

Then

$$\begin{aligned}\gamma_n^{**}-\gamma_{n+1}^{**}&=\log\left(n+\frac{3}{2}\right)-\log\left(n+\frac{1}{2}\right)-\frac{1}{n+1}\\&=\log\left(\frac{1+\frac{1}{2n+2}}{1-\frac{1}{2n+2}}\right)-\frac{1}{n+1}\\&=\frac{2}{3(2n+2)^3}+\frac{2}{5(2n+2)^5}+\frac{2}{7(2n+2)^7}+\ \cdots\ .\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n^{**}-\gamma_{n+1}^{**}&>\frac{2}{3(2n+2)^3}+\frac{2}{5(2n+2)^5}\\&>^*\frac{1}{24n(n+1)+\frac{51}{5}-\frac{1}{n(n+1)}}-\frac{1}{24(n+1)(n+2)+\frac{51}{5}-\frac{1}{(n+1)(n+2)}}\end{aligned}$$

for $n \geq 2$, and

$$\begin{aligned}\gamma_n^{**}-\gamma_{n+1}^{**}&<\frac{2}{3(2n+2)^3}+\frac{2}{5(2n+2)^5}+\frac{2}{7(2n+2)^7}\cdot\frac{1}{1-\frac{1}{(2n+2)^2}}\\&=\frac{2}{3(2n+1)^3}+\frac{2}{5(2n+1)^5}+\frac{2}{7(2n+1)(2n+2)^5(2n+3)}\\&<^*\frac{1}{24n(n+1)+\frac{51}{5}-\frac{3}{2n(n+1)}}-\frac{1}{24(n+1)(n+2)+\frac{51}{5}-\frac{3}{2(n+1)(n+2)}}.\end{aligned}$$

That is, for $n \geq 2$,

$$\begin{aligned}
&\frac{1}{24n(n+1)+\frac{51}{5}-\frac{1}{n(n+1)}}-\frac{1}{24(n+1)(n+2)+\frac{51}{5}-\frac{1}{(n+1)(n+2)}}\\
&\quad<\gamma_n^{**}-\gamma_{n+1}^{**}\\
&\quad<\frac{1}{24n(n+1)+\frac{51}{5}-\frac{3}{2n(n+1)}}-\frac{1}{24(n+1)(n+2)+\frac{51}{5}-\frac{3}{2(n+1)(n+2)}}.
\end{aligned}$$

If we sum this from n to ∞, we obtain

$$\frac{1}{24n(n+1)+\frac{51}{5}-\frac{1}{n(n+1)}}<\gamma_n^{**}-\gamma<\frac{1}{24n(n+1)+\frac{51}{5}-\frac{3}{2n(n+1)}},$$

for $n \geq 2$, which is (1.3).

3. Corollaries

We have

$$\begin{aligned}
\left(n+\frac{1}{2}\right)^2(\gamma_n^{**}-\gamma)&<\frac{\left(n+\frac{1}{2}\right)^2}{24n(n+1)+\frac{51}{5}-\frac{3}{2n(n+1)}}\\
&<^*\frac{\left(n+\frac{3}{2}\right)^2}{24(n+1)(n+2)+\frac{51}{5}-\frac{1}{(n+1)(n+2)}}\\
&<\left(n+\frac{3}{2}\right)^2(\gamma_{n+1}^{**}-\gamma)
\end{aligned}$$

and

$$\begin{aligned}
\left(\left(n+\frac{1}{2}\right)^2+\frac{1}{2}\right)(\gamma_n^{**}-\gamma)&>\frac{\left(\left(n+\frac{1}{2}\right)^2+\frac{1}{2}\right)}{24n(n+1)+\frac{51}{5}-\frac{1}{n(n+1)}}\\
&>^*\frac{\left(\left(n+\frac{3}{2}\right)^2+\frac{1}{2}\right)}{24(n+1)(n+2)+\frac{51}{5}-\frac{3}{2(n+1)(n+2)}}\\
&>\left(\left(n+\frac{3}{2}\right)^2+\frac{1}{2}\right)(\gamma_{n+1}^{**}-\gamma),
\end{aligned}$$

as claimed.

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MSC2010: 40A25, 33-01, 33E99, 33F05, 65D20

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