BEATTY SEQUENCES AND WYTHOFF SEQUENCES, GENERALIZED

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Abstract. Joint rankings of certain sets yield sequences called lower and upper s-Wythoff sequences. These generalizations of the classical Wythoff sequences include pairs of complementary Beatty sequences, both nonhomogeneous and homogeneous. There is a unique sequence Ψ such that the Ψ-Wythoff sequence of Ψ is Ψ. Finally, the Beatty discrepancy of a certain form of complementary equation is determined.

1. Introduction

Two well-known sequences associated with the golden ratio $\tau = (1+\sqrt{5})/2$ are the lower and upper Wythoff sequences:

$$(\lfloor n\tau \rfloor) = (1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, 27, \ldots),$$
$$(\lfloor n\tau^2 \rfloor) = (\lfloor n\tau \rfloor + n) = (2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, \ldots).$$

These Beatty sequences are indexed as A000201 and A001950 in [14], where many properties and references are given.

In many settings, *Beatty sequence* means a sequence of the form $(\lfloor nu \rfloor)$. Such sequences occur in complementary pairs, $(\lfloor nu \rfloor)$ and $(\lfloor nv \rfloor)$, where u is an irrational number greater than 1 and $v = u/(u-1)$. Here, however, we apply *Beatty sequence* more generally: a sequence of the form $(\lfloor nu + h \rfloor)$, where $u > 1$ and $1 \leq u + h$; elsewhere ([2, 3, 5, 12]), if $h \neq 0$, the sequence $(\lfloor nu + h \rfloor)$ is called a nonhomogeneous Beatty sequence.

Consider the following procedure for generating the classical Wythoff sequences. Write N in a row, write 1 at the beginning of a second row, and 2 at the beginning of a third row. Then generate row 2, labeled a, and row 3, labeled b, by taking $a(n)$ to be the least number missing from the set

$$\{a(1), a(2), \ldots, a(n-1), b(1), b(2), \ldots, b(n-1)\} \tag{1.1}$$

and $b(n) = n + a(n)$. The rows appear as follows:

n:	1	2	3	4	5	6	7	8	...
a:	1	3	4	6	8	9	11	12	...
b:	2	5	7	10	13	15	18	20	... .

The generalization indicated by the title stems from replacing N by an arbitrary nondecreasing sequence s of positive integers and putting $b(n) = s(n) + a(n)$, where $a(n)$ is given by (1.1). We call the resulting complementary sequences a and b the *lower* and *upper s-Wythoff sequences*.

In Section 2, formulas for various Beatty sequences are derived. In Section 3, we formulate s-Wythoff sequences for certain arithmetic sequences s. In Section 4, the procedure used to define s-Wythoff sequences is iterated, resulting in unique lower and upper limiting sequences. In Section 5, the notion of Beatty discrepancy is applied to certain s-Wythoff sequences.

Historical notes are of interest. The term *Beatty sequence* stems from a 1926 problem proposal, but Beatty's theorem – that the pairs of sequences are complementary – was known as early as 1894 by John William Strutt (Lord Rayleigh) [13]. The term *Wythoff sequence* stems from the winning pairs $(a(n), b(n))$ for the Wythoff game [17]. Aviezri Fraenkel and others [2, 4, 6, 7, 8, 10] have studied Beatty sequences and generalizations of the Wythoff game, some of which have winning pairs $(a(n), b(n))$ in which a and b are s-Wythoff sequences for various choices of s.

2. Joint Ranking of Two Sets

Suppose that u is a real number greater than 1, not necessarily irrational, and let $v = u/(u-1)$. Note that $u < v$ if and only if $u < 2$, and $u = v$ if and only if $u = 2$. We assume that $1 < u < v$, and if c is a real number for which the sets

$$S_1 = \{\frac{i}{u} + c : i \geq 1\} \quad \text{and} \quad S_2 = \{\frac{j}{v} : j \geq 1\} \tag{2.1}$$

are disjoint, we call (u, c) a *regular pair*. Suppose that the numbers in $S_1 \cup S_2$ are jointly ranked. Let $a(n)$ be the rank of $n/u + c$ and $b(n)$ the rank of n/v. Obviously, every positive integer is in exactly one of the sequences $a = (a(n))$ and $b = (b(n))$. In Theorem 1, we formulate $a(n)$ and $b(n)$ in terms of n, u, v, and c.

Theorem 1. *Suppose that (u, c) is a regular pair. Then the complementary joint-rank sequences $a(n)$ and $b(n)$ are given by*

$$a(n) = \begin{cases} n & \text{if } n \leq (1+cu)/(u-1) \\ \lfloor nu - cu \rfloor & \text{if } n > (1+cu)/(u-1) \end{cases}$$

$$b(n) = \lfloor nv + cv \rfloor,$$

for $n \geq 1$.

Proof. Clearly the number of numbers j for which $j/v \leq n/v$ is n. To find the number of numbers i satisfying $i/u + c \leq n/v$, first note that the inequality must be strict since $S_1 \cap S_2$ is empty, so that we seek the number of i such that

$$i < nu/v - cu,$$

or equivalently, $i < -n + un - cu$, since $u/v = u - 1$. If $-n + un - cu \leq 1$, the number of such i is zero; otherwise, the number is $\lfloor -n + nu - cu \rfloor$. Thus, the rank of n/v is the number $a(n)$ as stated. The same argument, together with the hypothesis that $u < v$, shows that the rank of $n/u + c$ is $b(n)$. □

The condition that $n > (1 + cu)/(u - 1)$ in Theorem 1 ensures that the sequences a and b are Beatty sequences if the condition holds for $n = 1$. This observation leads directly to the following corollary.

Corollary 1. *If (u, c) is a regular pair and $1 - 2/u < c \leq 1 - 1/u$, then the sequences a and b in Theorem 1 are complementary Beatty sequences: $a(n) = \lfloor nu - cu \rfloor$ and $b(n) = \lfloor nv + cv \rfloor$.*

Next, suppose that $a(n) = \lfloor nu + h \rfloor$ is a Beatty sequence. We wish to formulate its complement using Theorem 1. Specifically, we wish to find conditions on u and h under which there is a number h' such that the sequence given by $b(n) = \lfloor nv + h' \rfloor$ is the complement of a. In order to match $\lfloor nu + h \rfloor$ and $\lfloor nv + h' \rfloor$ to $\lfloor nu - cu \rfloor$ and $\lfloor nv + cv \rfloor$, respectively, we take $c = -h/u$ and $h' = -hv/u = h - hv$. The result is stated here as a second corollary.

Corollary 2. *Suppose that $u > 1$, and let $v = u/(u-1)$. Suppose that h is a number such that $1 - u \leq h < 2 - u$ and the sets $\{i/u - h : i \geq 1\}$ and $\{j/v : j \geq 1\}$ are disjoint. Let a be the sequence given by $a(n) = \lfloor nu + h \rfloor$ and let b be the complement of a. Then*

$$b(n) = \lfloor nv + h - hv \rfloor$$

for $n \geq 1$.

3. Generalized Wythoff Sequences

Suppose that $s = (s(n))$ is a nondecreasing sequence of positive integers. Define $a(1) = 1$, $b(1) = 1$, and for $n \geq 2$, define

$$\begin{aligned} a(n) &= \text{mex}\{a(1), a(2), \ldots, a(n-1), b(1), b(2), \ldots, b(n-1)\}; \\ b(n) &= s(n) + a(n). \end{aligned}$$

(The notation mex S, for *minimal excludant* (of a set S), means the least positive integer not in S; see the preprint of Fraenkel and Peled, *Harnessing the Unwieldy MEX Function*, downloadable from [9].) In the special case that $s(n) = n$ for all $n \geq 1$, the sequences a and b are the lower and upper Wythoff sequences, as in Section 1. In general, we call a the *lower s-Wythoff sequence* and b the *upper s -Wythoff sequence*. In this section, we shall prove that these are Beatty sequences when s is an arithmetic sequence of the form $s(n) = kn - w$, where k is a nonnegative integer and $w \in \{-1, 0, 1, 2, 3, \ldots, n-1\}$.

Example 1. *If s is the constant sequence given by $s(n) = 1$ for $n \geq 1$, then $a(n) = 2n - 1$ and $b(n) = 2n$ for every $n \geq 1$.*

Example 2. *If $s(n) = 2n$, then $a(n) = \lfloor \sqrt{2}n \rfloor$ and $b(n) = 2n + a(n)$ for every $n \geq 1$, a pair of homogeneous Beatty sequences.*

Example 3. *If $s(n) = n + 1$, then $a(n) = \lfloor \tau(n + 2 - \sqrt{5}) \rfloor$ and $b(n) = \lfloor \tau^2(n + 2 - \sqrt{5}) \rfloor$, a pair of Beatty sequences (A026273 and A026274 in* [14]), as in the next lemma.

Lemma 1. *Suppose that $s(n) = kn - w$, where $k \geq 1$ and $-1 \leq w \leq k - 1$. Let $d = \sqrt{k^2 + 4}$. The sequences*

$$A(n) = \left\lfloor \frac{d + 2 - k}{2} (n + \frac{w}{d + 2}) \right\rfloor \tag{3.1}$$

$$B(n) = \left\lfloor \frac{d + 2 + k}{2} (n - \frac{w}{d + 2}) \right\rfloor \tag{3.2}$$

are complementary.

Proof. In order to apply Corollary 1 to (3.1) and (3.2), let $u = (d+2-k)/2$ and $c = -w/(d+2)$, and let S_1 and S_2 be as in (2.1) with $v = u/(u-1)$. To see that S_1 and S_2 are disjoint, suppose for some i and j that $i/u + c = j/v$. In order to express j in a certain manner, note that

$$\begin{aligned} c &= -\frac{w}{2 + \sqrt{4 + k^2}}, \\ v &= \frac{(2 + k + \sqrt{4 + k^2})}{2}, \\ cv &= \frac{(2 - k - \sqrt{4 + k^2})}{2k} w, \end{aligned}$$

so that

$$j = \frac{iv}{u} + cv = \frac{w}{k} + \frac{(ik - w)(k + \sqrt{4 + k^2})}{2k}. \tag{3.3}$$

However, $\sqrt{4 + k^2}$ is irrational for all $k \geq 1$, so that the right-hand side of (3.3) is not an integer, proving that S_1 and S_2 are disjoint. By Corollary 1, the sequences (3.1) and (3.2) are a pair of complementary Beatty sequences. □

Theorem 2. *Suppose that $s(n) = kn - w$, where $k \geq 1$ and $-1 \leq w \leq k - 1$. Let $d = \sqrt{k^2 + 4}$, and let a and b be the lower and upper s-Wythoff sequences. Then $a = A$ and $b = B$, where A and B are given by (3.1) and (3.2).*

Proof. Clearly, $A(1) = a(1)$, and it is easy to check that $B(n) = kn - w + A(n)$ for all n, so that B and b arise from $B = s + A$ and $b = s + a$. Therefore, all we need to do is prove that if $n \geq 2$ and

$$m = \operatorname{mex}\{A(1), A(2), \ldots, A(n-1), B(1), B(2), \ldots, B(n-1)\},$$

then $m = A(n)$, but this is a direct consequence of Lemma 1. □

4. Limiting Sequences

Let Ψ denote the sequence

$$A003159 = (1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, \ldots)$$

in the Encyclopedia of Integer Sequences [14]; $\Psi(n)$ is then the nth positive integer whose binary representation ends in an even number of 0's. The complement of Ψ is the sequence $\Lambda = A036554 = 2 * A003159$ of numbers whose binary representation ends in an odd number of 0's. We shall prove that these two sequences are left fixed by the algorithm used to form s-Wythoff sequences. Then we shall prove that they are the unique limiting sequences when the procedure is iterated. (To say that $\lim_{m\to\infty} a_m = \Psi$ means that for every $H > 0$ there exists M such that if $m > M$, then $a_m(h) = \Psi(h)$ for all $h \leq H$.)

Theorem 3. *There exists a unique sequence Ψ such that the lower Ψ-Wythoff sequence of Ψ is Ψ.*

Proof. Suppose that $f = (f(n))$ is a sequence such that the lower f-Wythoff sequence of f is f. Let g be the upper f-Wythoff sequence. Clearly $g = 2f$. Since $f(1) = 1$, we have $g(1) = 2$, so that

$$f(2) = \operatorname{mex}\{f(1), g(1)\} = 3 \quad \text{and} \quad f(2) = 6.$$

As an inductive step, suppose for arbitrary $n \geq 2$ that $f(i)$ is uniquely determined for $i \leq n-1$. Let

$$T_{n-1} = \{1, 3, \ldots, f(n-1), 2, 6, \ldots, 2f(n-1)\}.$$

Then $\operatorname{mex}(T_{n-1})$ is given by one of two cases: $f(n) = f(n-1)+1$ if this number is not in T_{n-1} or else $f(n) = f(n-1) + 2$ since neighboring terms of the set $\{2, 6, \ldots, 2f(n-1)\}$ necessarily differ by at least 2. In both cases, $f(n)$ and hence $g(n)$ are uniquely determined. □

Henceforth we shall refer to Ψ and Λ as the lower and upper invariant Wythoff sequences. The next result indicates the special role played by these two sequences under iterations.

Theorem 4. *Suppose that s is a nondecreasing sequence in N. Let a_1 and b_1 be the lower and upper s-Wythoff sequences, respectively. Let $s_1 = a_1$, and let a_2 and b_2 be the lower and upper s_1-Wythoff sequences. Inductively, for $m \geq 2$, let $s_{m-1} = a_{m-1}$, and let a_m and b_m be the lower and upper s_{m-1}-Wythoff sequences. Then $\lim_{m\to\infty} a_m$ exists and is the lower invariant Wythoff sequence Ψ.*

Proof. As a first induction step, note that $s_1(1) = 1 = \Psi(1)$ even if $s(1) > 1$. As an induction hypothesis, suppose for $m \geq 1$ and $n \geq$ that $s_m(h) = a_m(h) = \Psi(h)$ for $h = 1, 2, \ldots, n$. Then

$$\begin{aligned} a(n+1) &= \operatorname{mex}\{a(1), a(2), \ldots, a(n-1), b(1), b(2), \ldots, b(n-1)\} \\ &= \operatorname{mex}\{\Psi(1), \Psi(2), \ldots, \Psi(n-1), \Lambda(1), \Lambda(2), \ldots, \Lambda(n-1)\} \end{aligned}$$

by the induction hypothesis, so that $a(n+1) = \Psi(n+1)$ by Theorem 3. Consequently, by induction, $\lim_{m\to\infty} a_m = \lim_{m\to\infty} s_m = \Psi$. □

5. Beatty Discrepancy

The notion of the Beatty discrepancy of a complementary equation is introduced in [14] at A138253. In this section we shall determine the Beatty discrepancy of certain equations of the form $b(n) = s(n) + a(n)$. We begin with definitions. Quoting from [11]: "Under the assumption that sequences a and b partition the sequence $N = (1, 2, 3, \ldots)$ of positive integers, the designation *complementary equations* applies to equations such as $b(n) = a(a(n)) + 1$ in much the same way that the designations *functional equations*, *differential equations*, and *Diophantine equations* apply elsewhere. Indeed, complementary equations can be regarded as a class of Diophantine equations."

Now suppose that a and b are solutions of a complementary equation $f(a, b) = 0$ and that the numbers $r = \lim_{n\to\infty} a(n)$ and $s = \lim_{n\to\infty} b(n)$ exist. Let $\alpha(n) = \lfloor rn \rfloor$ and $\beta(n) = \lfloor sn \rfloor$ for $n \geq 1$, so that α and β are a pair of complementary Beatty sequences. The Beatty discrepancy of the equation $f(a, b) = 0$ is the sequence $D = (D(n))$ defined by $D(n) = f(\alpha, \beta)$.

Theorem 5. *Suppose that $s(n) = kn - w$, where $k \geq 1$ and $w \in \{-1, 0, 1, 2, 3, \ldots, n-1\}$. Let $d = \sqrt{k^2+4}$, and let a and b be the lower and upper s-Wythoff sequences. Then the Beatty discrepancy of the equation $b = s + a$ is the constant sequence given by $D(n) = w$.*

Proof. Using A and B as in (3.1) and (3.2), we have

$$\begin{aligned} D(n) &= \left\lfloor \frac{d+2+k}{2} n \right\rfloor - (kn - w) - \left\lfloor \frac{d+2-k}{2} n \right\rfloor \\ &= \left\lfloor \frac{d+2-k}{2} n \right\rfloor + \lfloor kn \rfloor - kn + w - \left\lfloor \frac{d+2-k}{2} n \right\rfloor, \end{aligned}$$

so that $D(n) = w$ for all n. □

6. Concluding Comments

The Online Encyclopedia of Integer Sequences [14] includes several s-Wythoff sequences. For a guide to these and a *Mathematica* program for generating them, see A184117. In Theorems 1 and 5, the seed sequence s is an arithmetic sequence. It seems likely that these theorems can be generalized to cover a much wider class of nearly linear sequences.

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MSC2010: 11B85

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