FIBONACCI MEETS HOFSTADTER

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ABSTRACT. The Hofstatder Q sequence is defined by the recurrence relation $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$, with the initial conditions $Q(1) = Q(2) = 1$. Here we prove that other initial conditions can be used that cause the recurrence relation to generate the Fibonacci sequence.

1. Introduction

The term "meta-Fibonacci", first introduced in [5], has occurred in the title of a number of papers published in recent decades in the mathematical literature [4, 9, 10, 13, 1]. However, these papers scarcely mention the Fibonacci numbers! Why then, is that term used? A typical meta-Fibonacci recurrence relation is given below.

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)), \text{ with } Q(1) = Q(2) = 1. \tag{1.1}$$

Now the reason is apparent; it is because the recurrence for $Q(n)$ involves the addition of two terms, one containing $Q(n-1)$ and the other containing $Q(n-2)$. The sequence of numbers $Q(1), Q(2), Q(3), \ldots$ is very different from the Fibonacci sequence. For example, the numbers do not exhibit exponential growth; in fact, we seem to need $Q(n) < n$ for the recurrence to be well-defined. Nor are they monotone. The numbers generated by (1.1) are known as "Hofstadter's Q sequence," named after a sequence first introduced in [11]. Very little is known about this sequence, including whether it is well-defined for all $n > 1$.

If one changes the initial conditions to $Q(1) = 3$, $Q(2) = 2$, $Q(3) = 1$, then the resulting sequence is quasi-periodic with a quasi-period of 3. More precisely $Q(3k+1) = 3$, $Q(3k+2) = 3k+2$, and $Q(3k) = 3k-2$. This example was discovered by Golomb [8], and seems to be the only solved case of the Hofstadter Q recursion. There are some other nested recursions that give rise to the Fibonacci numbers, but these are not meta-Fibonacci; see Barbeau and Tanny [2], [3].

One way to make the Q sequence well-defined is to simply specify the initial values of Q for all $n < 1$, for example, that $Q(n) = 0$ if $n < 1$. We adopt that strategy here, and then determine initial conditions so that the Fibonacci sequence occurs in some natural way.

2. Finding Fibonacci Buried in Hofstadter

Define the Fibonacci sequence by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ if $n > 2$. Introduce the notation $P(m, j) := Q(3m + j)$ where (typically) $0 \leq j < 3$.

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Now suppose that the following formal solution to (1.1) holds for some unknown increasing positive sequence of integers $\alpha_0, \alpha_1, \alpha_2, \ldots$. What properties must α_m have?

$$P(m,j) = \begin{cases} 0 & \text{if } 3m+j<0 \\ 3 & \text{if } j=0 \\ 6 & \text{if } j=1 \\ \alpha_m & \text{if } j=2. \end{cases} \tag{2.1}$$

We inductively check (1.1), first for $j=0$, and later for $j=1,2$.

$$\begin{aligned} P(m,0) &= Q(3m) \\ &= Q(3m - Q(3(m-1)+2)) + Q(3m - Q(3(m-1)+1)) \\ &= Q(3m - P(m-1,2)) + Q(3m - P(m-1,1)) \\ &= P(m, 0 - P(m-1,2)) + P(m, 0 - P(m-1,1)) \\ &= P(m, -\alpha_{m-1}) + P(m,-6) \\ &= 0 + P(m-2,0) \\ &= 3. \end{aligned}$$

The last two equalities hold if $3m - \alpha_{m-1} < 0$ and $m-2 \geq 0$. We will now do a similar calculation for $j=1$ and $j=2$, skipping some of the more obvious steps.

$$\begin{aligned} P(m,1) &= P(m, 1 - P(m,0)) + P(m, 1 - P(m-1,2)) \\ &= P(m, 1-3) + P(m, 1-\alpha_{m-1}) \\ &= P(m-1,1) + 0 \\ &= 6. \end{aligned}$$

The last two equalities hold if $m-1 \geq 0$ and $3m+1-\alpha_{m-1} < 0$.

$$\begin{aligned} P(m,2) &= P(m, 2 - P(m-1,1)) + P(m, 2 - P(m-1,0)) \\ &= P(m,-4) + P(m,-1) \\ &= P(m-2,2) + P(m-1,2) \\ &= \alpha_{m-2} + \alpha_{m-1}. \end{aligned}$$

The equalities hold if $m-2 \geq 0$ and $\alpha_m = \alpha_{m-2} + \alpha_{m-1}$. Note that the Fibonacci recurrence relation has (somewhat mysteriously) appeared. If we satisfy the various inequalities that have occurred and supply the appropriate initial conditions, then we should be able to generate the Fibonacci sequence.

All the constraints in the three cases are subsumed by $m \geq 2$ and $3m+1 < \alpha_{m-1}$. When $m=1$ the second constraint is $4 < \alpha_0$, and when $m=2$ it is $7 < \alpha_1$. The smallest consecutive Fibonacci numbers that satisfy these constraints are $\alpha_0 = 5$ and $\alpha_1 = 8$. Thus if we make $P(0,2) = 5 = F_5$ and $P(1,2) = 8 = F_6$ as initial conditions (along with $P(0,0) = P(1,0) = 3$ and $P(0,1) = P(1,1) = 6$), then for $m \geq 2$, successive α_m will be successive Fibonacci numbers. In conclusion we state the following theorem.

Theorem 2.1. *If $Q(n)$ is defined by the following recurrence relation*

$$Q(n) = \begin{cases} 0 & \text{if } n < 0 \\ 3 & \text{if } n = 0, 3 \\ 6 & \text{if } n = 1, 4 \\ 5 & \text{if } n = 2 \\ 8 & \text{if } n = 5 \\ Q(n - Q(n-1)) + Q(n - Q(n-2)) & \text{if } n > 5 \end{cases} \tag{2.2}$$

then, for all $m \geq 0$,

$$Q(3m+2) = F_{m+5}.$$

3. Open Problems

The result given in this paper is meant to whet the reader's appetite. Below we list some questions that await further investigation.

- Both of the known quasi-periodic solutions to the Hofstadter recurrence have quasi-period 3. Are other quasi-periods possible?
- Are there other quasi-periodic solutions to the Hofstadter recurrence that yield Fibonacci numbers?
- Are there quasi-periodic solutions to other meta-Fibonacci recurrences, such as the Conolly or Tanny recurrences [5, 15]?
- The answer to the previous question is certainly affirmative, but one could ask, for example, whether every rational sequence occurs as the quasi-periodic solution to some meta-Fibonacci recurrence.

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