SERIES REPRESENTATIONS OF THETA FUNCTIONS IN TERMS OF A SEQUENCE OF POLYNOMIALS

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ABSTRACT. We derive series expansions for the Jacobi theta functions $\theta_j(q)$, $j = 2, 3, 4$, and for $\theta_3(z, q)$, all in terms of a certain sequence of sparse binomial-type polynomials. As consequences we obtain series identities involving second-order recurrence sequences and Chebyshev polynomials of the first kind.

1. INTRODUCTION

The Jacobi theta functions belong to the most important special functions in mathematics, with applications in analysis, number theory, and combinatorics. They are four interrelated quasi-doubly periodic functions in the complex variable z and also depend on the *nome* q, $|q| < 1$. For instance,

$$\theta_3(z,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz); \tag{1.1}$$

see, e.g., [5, Ch. 20] or [1, p. 508ff.] for this and the other functions. Of special interest are these functions at $z = 0$, namely

$$\theta_j(q) := \theta_j(0,q), \quad j = 2, 3, 4$$

(note that $\theta_1(0, q) = 0$). In particular, we have

$$\theta_2(q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} = 2q^{1/4}\sum_{n=1}^{\infty} q^{n(n-1)}, \tag{1.2}$$

$$\theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \qquad \theta_4(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \tag{1.3}$$

These last functions are especially useful in additive number theory. For example, by equating coefficients of powers of q it is easy to see that

$$\theta_3(q)^k = \sum_{n=0}^{\infty} r_k(n) q^n,$$

where $r_k(n)$ is the number of ways n can be written as a sum of k squares; see, e.g., [1, p. 506] for this and other similar relations.

It is the purpose of this paper to derive infinite series expansions for $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$, as well as for $\theta_3(z, q)$, all in terms of the special polynomials

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$$f_n(z) := \sum_{k=0}^{n} \binom{n}{k} z^{k(k-1)/2}. \tag{1.4}$$

Recently the authors [2] defined and used these polynomials in the following graph theoretical setting. An *independent set* of vertices of a (finite simple) graph is a subset of the vertices of the graph, no two of which are joined by an edge. Consider the complete graph K_n and assume that every edge may be deleted independently with equal probability $p = 1 - q$, $(0 < q < 1)$. Then the expected number of independent sets of a graph of order n is given by $f_n(q)$.

In [2] the authors study, among other things, the growth and asymptotic behavior of $f_n(x)$. For instance, it was shown that for fixed real x with $0 < x < 1$ we have asymptotically

$$\log f_n(x) \sim \frac{1}{2\log(1/x)} \log^2 n \quad \text{as} \quad n \to \infty. \tag{1.5}$$

The similarity of the right-hand side of (1.4) to the usual binomial expansion, and the special form of the exponents of z, make the polynomials $f_n(z)$ interesting objects to study in their own right. Therefore the authors investigated their algebraic and analytic properties in the forthcoming paper [3]; numerous results have been obtained, including the distribution of complex and negative real zeros.

In Section 2 we prove a lemma involving these polynomials, which will be the basis for all further results. Section 3 contains the main results and their proofs, and in Section 4 we derive a number of consequences.

2. A Basic Lemma

We begin our present study with an easy lemma. Throughout the remainder of this paper we have $z = q^2$ for a complex q with $|q| < 1$.

Lemma 2.1. *For complex q and t with $|q| < 1$, $|t| < 1$ we have*

$$\sum_{n=0}^{\infty} f_n(q^2)t^n = \frac{1}{1-t} \sum_{k=0}^{\infty} q^{k(k-1)} \left(\frac{t}{1-t}\right)^k. \tag{2.1}$$

Before proving this lemma, we make some remarks on the sizes of the values of $f_n(x)$. By the definition (1.4) we have for $|x| < 1$,

$$|f_n(x)| < \sum_{k=0}^{n} \binom{n}{k} = 2^n = f_n(1). \tag{2.2}$$

However, (1.5) implies that for any fixed x, $0 < x < 1$, we have

$$\lim_{n\to\infty} f_n(x)^{1/n} = \exp\left(\lim_{n\to\infty} \frac{1}{n} \log f_n(x)\right) = e^0 = 1, \tag{2.3}$$

in contrast to the upper bound.

Proof of Lemma 2.1. Let $\varepsilon > 0$ and suppose that $|q| \le 1 - \varepsilon$ and $|t| \le 1 - \varepsilon$. Since $|f_n(q^2)| \le f_n(|q|^2)$, the left-hand side of (2.1) is uniformly convergent by (2.3). Furthermore, since $|t/(1-t)| \le T_\varepsilon$ for all t with $|t| < 1 - \varepsilon$, where T_ε is some finite bound, we have

$$\left| q^{k(k-1)} \left(\frac{t}{1-t}\right)^k \right|^{1/k} \le |1-\varepsilon|^{k-1} T_\varepsilon \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore the right-hand side of (2.1) is also uniformly convergent, and the following operations are legitimate. Now, using the definition (1.4), the left-hand side of (2.1) becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}q^{k(k-1)}t^n = \sum_{k=0}^{\infty}q^{k(k-1)}\sum_{n=k}^{\infty}\binom{n}{k}t^n. \tag{2.4}$$

The inner sum on the right can be rewritten as

$$\sum_{n=k}^{\infty}\binom{n}{n-k}t^n = t^k\sum_{n=0}^{\infty}\binom{n+k}{n}t^n = t^k\frac{1}{(1-t)^{k+1}},$$

where we have used a well-known series evaluation; see, e.g., [4, (1.3)]. This, together with (2.4), gives (2.1), valid for $|t| \le 1-\varepsilon$. Since $\varepsilon > 0$ was arbitrary, [2] holds for all $|t| < 1$. □

3. The Main Results

We are now ready to state and prove the following representations.

Theorem 3.1. *For $|q| < 1$ we have*

$$\sum_{n=0}^{\infty}2^{-n}f_n(q^2) = 2 + q^{-1/4}\theta_2(q), \tag{3.1}$$

and for $|q| < \frac{1}{2}$,

$$\sum_{n=0}^{\infty}\frac{2q^n}{(1+q)^{n+1}}f_n(q^2) = 1 + \theta_3(q), \tag{3.2}$$

$$\sum_{n=0}^{\infty}\frac{2(-q)^n}{(1-q)^{n+1}}f_n(q^2) = 1 + \theta_4(q). \tag{3.3}$$

Proof. The identity (3.1) follows immediately from (2.1) and (1.2), by setting $t = \frac{1}{2}$. Next, let $t = \pm q/(1 \pm q)$. Then

$$\frac{t}{1-t} = \pm q \quad \text{and} \quad \frac{1}{1-t} = 1 \pm q,$$

and $|q| < \frac{1}{2}$ implies $|t| < 1$. So (2.1), together with both parts of (1.3), immediately gives (3.2) and (3.3). □

Next, we use the same method as before and derive a representation of $\theta_3(z,q)$, for $z \in \mathbb{R}$, in terms of the polynomials $f_n(q^2)$. The following result can be seen as representative of the other theta functions $\theta_j(z,q)$ which, by the way, can all be written in terms of $\theta_3(z,q)$.

Theorem 3.2. *For $|q| < \frac{1}{2}$ and $z \in \mathbb{R}$ we have*

$$\sum_{n=0}^{\infty}\left(e^{-2iz}\left(\frac{qe^{2iz}}{1+qe^{2iz}}\right)^{n+1} + e^{2iz}\left(\frac{qe^{-2iz}}{1+qe^{-2iz}}\right)^{n+1}\right)\frac{f_n(q^2)}{q} = 1 + \theta_3(z,q). \tag{3.4}$$

Proof. We use (2.1) with

$$t = \frac{qe^{\pm 2iz}}{1+qe^{\pm 2iz}}.$$

Since $z \in \mathbb{R}$ and $|q| < \frac{1}{2}$, we see that $|t| < 1$ so that (2.1) applies. Also, in the analogy to the proof of (3.2) and (3.3) we have

$$\frac{t}{1-t} = qe^{\pm 2iz} \quad \text{and} \quad \frac{1}{1-t} = 1 + qe^{\pm 2iz}.$$

We then get with (2.1),

$$\sum_{n=0}^{\infty} f_n(q^2) \frac{(qe^{\pm 2iz})^n}{(1+qe^{\pm 2iz})^{n+1}} = 1 + \sum_{k=1}^{\infty} q^{k^2} e^{\pm 2kiz}. \tag{3.5}$$

Finally, we add (3.5) for "+" and for "−"; then (1.1) immediately gives (3.4). □

4. Some Consequences

Theorem 3.2 is particularly suitable for deriving identities that involve second-order linear recurrence sequences. The following is a first example.

Corollary 1. *Let F_n be the nth Fibonacci number (with $F_0 = 0, F_1 = 1$). Then*

$$\frac{5}{2} \sum_{n=0}^{\infty} (-1)^n \frac{F_{n+1}}{2^{n+1}} f_n(\tfrac{-1}{5}) = \sum_{k=0}^{\infty} (-1)^k 25^{-k^2}. \tag{4.1}$$

Proof. We use (3.4) with $z = \pi/4$ and $q = i/\sqrt{5}$. Then $e^{\pm 2iz} = \pm i$, and we get

$$\frac{qe^{\pm 2iz}}{1+qe^{\pm 2iz}} = -\frac{1}{2}\left(\frac{1 \pm \sqrt{5}}{2}\right).$$

Now, using the well-known Binet formula for the Fibonacci numbers, namely

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \left(\tfrac{1-\sqrt{5}}{2}\right)^n\right), \tag{4.2}$$

we easily see that the left-hand side of (3.4) gives twice the left-hand side of (4.1). On the other hand, we use that fact that

$$\cos(2nz) = \cos(\tfrac{n\pi}{2}) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^k, & n = 2k. \end{cases}$$

Hence, by (1.1) we have

$$1 + \theta_3(\tfrac{\pi}{4}, \tfrac{i}{\sqrt{5}}) = 2 + 2\sum_{k=1}^{\infty} (-1)^k \left(\frac{i}{\sqrt{5}}\right)^{(2k)^2} = 2\sum_{k=0}^{\infty} (-1)^k 25^{-k^2},$$

which completes the proof. □

Apart from the occurrence of the Fibonacci numbers, the identity (4.1) is interesting because of the fact that the right-hand series converges extremely quickly, while the left-hand series does so very slowly. In fact, adding the left-hand side up to $n = 50$ gives an error of about 0.0035, and up to $n = 100$ the error is still about $0.5 \cdot 10^{-6}$.

This last proof shows that a large number of similar identities can be obtained from (4.1) by choosing different values of z and q, where $z = \pi/4$ is particularly convenient, while $z = 0$ recovers (3.2). We now state, without a detailed proof, another identity which is obtained by taking $z = \pi/4$. Here we choose $q = 1/\sqrt{10}$; in this case the analogue of the Binet formula (4.2) is

$$u_n := \frac{1}{i\sqrt{10}} \left((1+i\sqrt{10})^n - (1-i\sqrt{10})^n \right),$$

and the sequence u_n satisfies the recurrence

$$u_n = 2u_{n-1} - 11u_{n-2}, \quad \text{with} \quad u_0 = 0, u_1 = 2,$$

so that the next few terms are $4, -14, -72, 10, 812, 1514, -5904, \dots$.

Corollary 2. *Let the sequence* $\{u_n\}$ *be defined as above. Then*

$$5\sum_{n=0}^{\infty} \frac{u_{n+1}}{11^{n+1}} f_n(\tfrac{1}{10})) = \sum_{k=0}^{\infty} (-1)^k 100^{-k^2}. \tag{4.3}$$

A final application of (3.4) involves the Chebyshev polynomials of the first kind, $T_n(x)$, which can be defined by

$$T_n(x) := \cos(n\cos^{-1} x) = \frac{n}{2} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} (2x)^{n-2j}; \tag{4.4}$$

see, e.g., [5, Ch. 18].

Corollary 3. *Suppose that the real numbers* q *and* z *are related through the identity* $q = -1/(2\cos 2z)$*, with* $|q| < 1$*. Then*

$$\frac{2}{q} \sum_{n=0}^{\infty} (-1)^{n+1} T_{2n+1}(\tfrac{-1}{2q}) f_n(q^2) = 1 + \theta_3(z, q). \tag{4.5}$$

Proof. We use (3.4) with

$$q = \frac{-1}{e^{2iz} + e^{-2iz}} = \frac{-1}{2\cos(2z)}. \tag{4.6}$$

Then it is easy to see that

$$\frac{qe^{\pm 2iz}}{1 + qe^{\pm 2iz}} = -e^{\pm 4iz},$$

and the expression in square brackets in (3.4) becomes

$$\begin{aligned} (-1)^{n+1} \left(e^{(2n+1)2iz} + e^{-(2n+1)2iz} \right) &= (-1)^{n+1} 2\cos((2n+1)2z) \\ &= (-1)^{n+1} 2T_{2n+1}(\tfrac{-1}{2q}), \end{aligned}$$

where the second equality follows from (4.4) and (4.6). With (3.4) this immediately gives (4.5). □

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