SUBSEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS WITH GEOMETRIC SUBSCRIPTS

WENCHANG CHU AND NADIA N. LI

Abstract. For the bivariate Fibonacci and Lucas polynomials with their subscripts being of geometric series, we establish twelve interrelations and recurrence relations among which four equalities are the main results found recently by Kilic and Tan (2010).

1. Introduction and Preliminaries

For two indeterminate p and q subject to $\Delta = p^2 - 4q \neq 0$, define the bivariate Fibonacci and Lucas polynomials by

$$\begin{aligned} U_n(p,q) &= pU_{n-1}(p,q) - qU_{n-2}(p,q) \quad \text{for} \quad n > 1 \\ &\quad \text{with} \quad U_0(p,q) = 0 \quad \text{and} \quad U_1(p,q) = 1; \\ V_n(p,q) &= pV_{n-1}(p,q) - qV_{n-2}(p,q) \quad \text{for} \quad n > 1 \\ &\quad \text{with} \quad V_0(p,q) = 2 \quad \text{and} \quad V_1(p,q) = p. \end{aligned}$$

By means of the generating function method (cf. Wilf [9]), it is not hard to show the following Binet forms

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(p,q) = \alpha^n + \beta^n$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \qquad \text{and} \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}.$$

Throughout the paper, the two parameters p and q will be dropped in the notations $U_n := U_n(p,q)$ and $V_n := V_n(p,q)$ for the sake of brevity. Chu and Yan [3] derived several multiple convolution formulas for these polynomials.

These two sequences extend the classical Fibonacci and Lucas numbers, respectively by $U_n(1,-1) = F_n$ and $V_n(1,-1) = L_n$. For the latter, there exist the following curious relations

$$\begin{aligned} F_{3^{n+1}} &= 5F_{3^n}^3 - F_{3^n}, \\ L_{4^{n+1}} &= L_{4^n}^4 - 4L_{4^n}^2 + 2, \\ F_{5^{n+1}} &= 25F_{5^n}^5 - 25F_{5^n}^3 + 5F_{5^n}, \\ L_{6^{n+1}} &= L_{6^n}^6 - 6L_{6^n}^4 + 9L_{6^n}^2 - 2; \end{aligned}$$

where the first one was a problem proposed by Filipponi and solved by Terr [6]; while the remaining three relations appeared in the comment by Klamkin [6].

Inspired by these equalities, Kilic and Tan [7, 8] find interrelations and recurrence relations for U_m and V_m when the subscripts $\{m\}$ are of geometric series $\{k^n\}$. Their main results are three recurrence relations of the first order for U_{k^n} and V_{k^n} plus one polynomial expression of V_{k^n} in terms of U_{k^n} (cf. Corollaries 7–10), that are derived by combining the binomial theorem with the following two fundamental binomial relations, which can be found in Carlitz [2, Page 23] and Comtet [4, Section 4.9], respectively:

$$\frac{x^m - y^m}{x - y} = \sum_{0 \le k < m/2} (-1)^k \binom{m-k-1}{k} (xy)^k (x+y)^{m-2k-1}, \tag{1}$$

$$x^m + y^m = \sum_{0 \le k \le m/2} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (xy)^k (x+y)^{m-2k}. \tag{2}$$

The purpose of this paper is to show that these polynomial identities can be utilized to investigate systematically the interrelations concerning U_{k^n} and V_{k^n}. In the next section, by examining these two identities carefully, we find that they contain not only the aforementioned four relations due to Kilic and Tan [7, 8] as very particular instances, but also four other similar interrelations. Furthermore, we shall establish, in the third section, four quite unusual interrelations for U_{k^n} and V_{k^n} by employing Dougall's Theorem for terminating well–poised ${}_4F_3$-series in the evaluation of connection coefficients. Finally, the paper will end up with a collection of 22 curious polynomial identities involving the classical Fibonacci and Lucas numbers. In order to guarantee the accuracy of computations, all the equalities appearing in this paper have been verified through *Mathematica* commands.

2. Eight Easier Relations Among U_{k^n} and V_{k^n}

Letting $x = \alpha^n$ and $y = \beta^n$ in (1) and (2), we get directly the following relations for U_{mn} and V_{mn}:

$$U_{mn} = U_n \sum_{0 \le k < m/2} (-q^n)^k \binom{m-k-1}{k} V_n^{m-2k-1}, \tag{3}$$

$$V_{mn} = \sum_{0 \le k \le m/2} (-q^n)^k \frac{m}{m-k} \binom{m-k}{k} V_n^{m-2k}. \tag{4}$$

According to the parity of m, they can explicitly be displayed in the following lemma as four expressions in terms of V_n.

Lemma 1 ($m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$)**.**

$$U_{2mn} = U_n \sum_{k=1}^{m} (-q^n)^{m-k} \binom{m+k-1}{2k-1} V_n^{2k-1}, \tag{5}$$

$$U_{2mn+n} = U_n \sum_{k=0}^{m} (-q^n)^{m-k} \binom{m+k}{2k} V_n^{2k}, \tag{6}$$

$$V_{2mn} = \sum_{k=0}^{m} (-q^n)^{m-k} \frac{2m}{m+k} \binom{m+k}{2k} V_n^{2k}, \tag{7}$$

$$V_{2mn+n} = \sum_{k=0}^{m} (-q^n)^{m-k} \frac{1+2m}{1+m+k} \binom{1+m+k}{1+2k} V_n^{1+2k}. \tag{8}$$

Instead, there exist also four companion expressions in terms of U_n.

Lemma 2 ($m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$)**.**

$$U_{2mn} = V_n \sum_{k=1}^{m} q^{n(m-k)} \binom{m+k-1}{2k-1} \Delta^{k-1} U_n^{2k-1}, \tag{9}$$

$$U_{2mn+n} = \sum_{k=0}^{m} q^{n(m-k)} \frac{1+2m}{1+m+k} \binom{1+m+k}{1+2k} \Delta^{k} U_n^{2k+1}, \tag{10}$$

$$V_{2mn} = \sum_{k=0}^{m} q^{n(m-k)} \frac{2m}{m+k} \binom{m+k}{2k} \Delta^{k} U_n^{2k}, \tag{11}$$

$$V_{2mn+n} = V_n \sum_{k=0}^{m} q^{n(m-k)} \binom{m+k}{2k} \Delta^{k} U_n^{2k}. \tag{12}$$

They can be deduced from (1) and (2) in the following manner:

$$\begin{array}{llllll}
\text{Eq}(9): & m \to 2m, & x \to \alpha^n & \text{and} & y \to -\beta^n & \text{in} \quad (1), \\
\text{Eq}(10): & m \to 2m+1, & x \to \alpha^n & \text{and} & y \to -\beta^n & \text{in} \quad (2), \\
\text{Eq}(11): & m \to 2m, & x \to \alpha^n & \text{and} & y \to -\beta^n & \text{in} \quad (2), \\
\text{Eq}(12): & m \to 2m+1, & x \to \alpha^n & \text{and} & y \to -\beta^n & \text{in} \quad (1).
\end{array}$$

Among these eight relations just displayed, four of them labeled with (7), (8), (10), and (11) have been obtained by Kilic and Tan [8, Lemma 1]. Both Lemmas 1 and 2 are quite useful to derive expressions between the subsequences of U_{k^n} and V_{k^n} with the subscripts $\{k^n\}$ being a geometric series.

Proposition 3 ($n \to k^n$ and $m \to k/2$ in (9) with k being even)**.**

$$U_{k^{n+1}} = V_{k^n} \sum_{i=1}^{k/2} q^{k^n(k/2-i)} \binom{i-1+\frac{k}{2}}{2i-1} \Delta^{i-1} U_{k^n}^{2i-1}.$$

Proposition 4 ($n \to k^n$ and $m \to k/2$ in (5) with k being even)**.**

$$U_{k^{n+1}} = U_{k^n} \sum_{i=1}^{k/2} (-q^{k^n})^{k/2-i} \binom{i-1+\frac{k}{2}}{2i-1} V_{k^n}^{2i-1}.$$

Proposition 5 ($n \to k^n$ and $m \to (k-1)/2$ in (6) with k being odd)**.**

$$U_{k^{n+1}} = U_{k^n} \sum_{i=0}^{(k-1)/2} (-q^{k^n})^{(k-1-2i)/2} \binom{(k-1)/2+i}{2i} V_{k^n}^{2i}.$$

Proposition 6 ($n \to k^n$ and $m \to (k-1)/2$ in (12) with k being odd)**.**

$$V_{k^{n+1}} = V_{k^n} \sum_{i=0}^{(k-1)/2} q^{k^n(k-1-2i)/2} \binom{(k-1)/2+i}{2i} \Delta^{i} U_{k^n}^{2i}.$$

As immediate consequences, we can also recover the following corollaries, that are the main theorems due to Kilic and Tan [8, Theorems 1–4], who found them by manipulating the binomial expansions for $U_{k^n}^k$ and $V_{k^n}^k$.

Corollary 7 ($n \to k^n$ and $m \to k/2$ in (11) with k being even)**.**

$$V_{k^{n+1}} = \sum_{i=0}^{k/2} q^{k^n(k/2-i)} \frac{2k}{k+2i} \binom{k/2+i}{2i} \Delta^i U_{k^n}^{2i}.$$

Corollary 8 ($n \to k^n$ and $m \to k/2$ in (7) with k being even)**.**

$$V_{k^{n+1}} = \sum_{i=0}^{k/2} (-q^{k^n})^{k/2-i} \frac{2k}{k+2i} \binom{k/2+i}{2i} V_{k^n}^{2i}.$$

Corollary 9 ($n \to k^n$ and $m \to (k-1)/2$ in (10) with k being odd)**.**

$$U_{k^{n+1}} = \sum_{i=0}^{(k-1)/2} q^{k^n(k-1-2i)/2} \frac{2k}{1+k+2i} \binom{i+\frac{k+1}{2}}{1+2i} \Delta^i U_{k^n}^{1+2i}.$$

Corollary 10 ($n \to k^n$ and $m \to (k-1)/2$ in (8) with k being odd)**.**

$$V_{k^{n+1}} = \sum_{i=0}^{(k-1)/2} (-q^{k^n})^{(k-1-2i)/2} \frac{2k}{k+2i+1} \binom{(k+1)/2+i}{2i+1} V_{k^n}^{2i+1}.$$

We remark that the special case $p = -q = 1$ of Corollary 9 was previously obtained by Filipponi [5, Proposition 4]

$$F_{k^{n+1}} = \sum_{i=0}^{(k-1)/2} (-1)^{i+(k-1)/2} \frac{2k \cdot 5^i}{1+k+2i} \binom{i+\frac{k+1}{2}}{1+2i} F_{k^n}^{1+2i}. \tag{13}$$

3. Four Further Relations Among U_{k^n} and V_{k^n}

By means of binomial expansions for $U_{k^n}^k$ and $V_{k^n}^k$ as done in Kilic and Tan [8], we shall establish in this section, four further relations between U_{k^n} and V_{k^n}. In order to evaluate the binomial sums of connection coefficients, we shall utilize Dougall's Theorem [1, Section 4.4, pp. 27] for terminating well-poised series

$$\begin{aligned} \frac{(1+a)_n}{(1+a-c)_n} &= {}_4F_3\left[\begin{matrix} a,\ 1+a/2, & c, & -n \\ & a/2,\ 1+a-c, & 1+a+n \end{matrix} \,\middle|\, -1\right] \\ &= \sum_{j=0}^{n} \frac{a+2j}{a} \frac{(a)_j (c)_j (-n)_j}{j!(1+a-c)_j (1+a+n)_j} (-1)^j \end{aligned} \tag{14}$$

where the shifted factorial is given by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{with} \quad n \in \mathbb{N}$$

and the classical hypergeometric series reads as

$${}_{1+\ell}F_\ell\left[\begin{matrix} a_0, a_1, \ldots, a_\ell \\ c_1, \ldots, c_\ell \end{matrix} \,\middle|\, x\right] = \sum_{k \geq 0} \frac{(a_0)_k (a_1)_k \cdots (a_\ell)_k}{k!(c_1)_k \cdots (c_\ell)_k} x^k.$$

According to the binomial theorem, we can manipulate $V_{k^n}^k$ as

$$\begin{aligned} V_{k^n}^k &= (\alpha^{k^n}+\beta^{k^n})^k = \sum_{j=0}^{k}\binom{k}{j}\alpha^{jk^n}\beta^{(k-j)k^n} \\ &= \chi\, k \equiv 0 \pmod 2 \binom{k}{k/2} q^{k^{n+1}/2} + \sum_{0\le j<k/2}\binom{k}{j} q^{jk^n} V_{(k-2j)k^n}. \end{aligned} \tag{15}$$

When k is even, the last relation can be restated as

$$V_{k^n}^k = V_{k^{n+1}} + \binom{k}{k/2} q^{k^{n+1}/2} + \sum_{j=1}^{k/2-1}\binom{k}{j} q^{jk^n} V_{(k-2j)k^n}. \tag{16}$$

Applying the relation (11) to the last $V_{(k-2j)k^n}$ and then simplifying the double sum expression, we get

$$\begin{aligned} V_{k^n}^k &= V_{k^{n+1}} + \binom{k}{k/2} q^{k^{n+1}/2} + 2\sum_{j=1}^{k/2-1}\binom{k}{j} q^{k^{n+1}/2} \\ &\quad + \sum_{j=1}^{k/2-1}\binom{k}{j}\sum_{i=1}^{k/2-j}\Delta^i U_{k^n}^{2i} q^{k^n(k/2-i)}\binom{\frac{k}{2}-j+i}{2i}\frac{k-2j}{\frac{k}{2}-j+i} \\ &= V_{k^{n+1}} + (2^k-2) q^{k^{n+1}/2} \\ &\quad + \sum_{i=1}^{k/2-1}\Delta^i U_{k^n}^{2i} q^{k^n(k/2-i)}\sum_{j=1}^{k/2-i}\binom{k}{j}\binom{\frac{k}{2}-j+i}{2i}\frac{k-2j}{\frac{k}{2}-j+i}. \end{aligned}$$

Observing that (14) can be used to evaluate the following binomial sum

$$\begin{aligned} &\sum_{j=0}^{k/2-i}\binom{k}{j}\binom{\frac{k}{2}-j+i}{2i}\frac{k-2j}{\frac{k}{2}-j+i} \\ = \quad & \frac{2k}{2i+k}\binom{i+k/2}{2i}\sum_{j=0}^{k/2-i}\frac{k-2j}{k}\frac{(-k)_j(i-k/2)_j}{j!(1-i-k/2)_j}(-1)^j \\ = \quad & \frac{2k}{2i+k}\binom{i+k/2}{2i}{}_3F_2\left[\begin{matrix}-k,\ 1-k/2,\ i-k/2 \\ -k/2,\ 1-i-k/2\end{matrix}\ \Big|\ -1\right] \\ = \quad & \frac{2k}{2i+k}\binom{i+k/2}{2i}\frac{(i+k/2)_{k/2-i}}{(i+1/2)_{k/2-i}} = 2^{k-2i}\binom{\frac{k}{2}}{i} \end{aligned}$$

we derive the following expression of $V_{k^{n+1}}$ in terms of U_{k^n}, which may be considered as a variant of Corollary 7.

Theorem 11 ($n,\ k \in \mathbb{N}$ with k being even)**.**

$$V_{k^{n+1}} = V_{k^n}^k - \sum_{i=0}^{k/2-1}\Delta^i U_{k^n}^{2i} q^{k^n(k/2-i)}\left\{2^{k-2i}\binom{\frac{k}{2}}{i} - \frac{2k}{k+2i}\binom{\frac{k}{2}+i}{2i}\right\}.$$

When k is odd, the equation (15) can be reformulated as

$$V_{k^n}^k = V_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}} \binom{k}{j} q^{jk^n} V_{(k-2j)k^n}. \tag{17}$$

Applying (12) to the last $V_{(k-2j)k^n}$, we get the double sum expression

$$V_{k^n}^k = V_{k^{n+1}} + V_{k^n} \sum_{i=0}^{(k-3)/2} q^{k^n(\frac{k-1}{2}-i)} \Delta^i U_{k^n}^{2i} \sum_{j=1}^{\frac{k-1}{2}-i} \binom{k}{j} \binom{\frac{k-1}{2}-j+i}{2i}.$$

The last binomial sum can be evaluated by (14) as follows:

$$\begin{aligned}\sum_{j=0}^{\frac{k-1}{2}-i} \binom{k}{j} \binom{\frac{k-1}{2}-j+i}{2i} &= \binom{\frac{k-1}{2}+i}{2i} \sum_{j=0}^{\frac{k-1}{2}-i} (-1)^j \frac{(-k)_j (i-\frac{k-1}{2})_j}{j!(-i-\frac{k-1}{2})_j} \\ &= \binom{\frac{k-1}{2}+i}{2i} {}_2F_1 \left[\begin{matrix} -k, i-\frac{k-1}{2} \\ -i-\frac{k-1}{2} \end{matrix} \,\middle|\, -1 \right] \\ &= \binom{\frac{k-1}{2}+i}{2i} \frac{(i+\frac{k+1}{2})_{\frac{k-1}{2}-i}}{(i+\frac{1}{2})_{\frac{k-1}{2}-i}} = 2^{k-2i-1} \binom{\frac{k-1}{2}}{i}.\end{aligned}$$

This yields the following polynomial representation, which differs substantially from Proposition 6.

Theorem 12 ($n,\ k \in \mathbb{N}$ with k being odd)**.**

$$V_{k^{n+1}} = V_{k^n}^k - V_{k^n} \sum_{i=0}^{(k-3)/2} \Delta^i q^{k^n(\frac{k-1}{2}-i)} U_{k^n}^{2i} \left\{ 2^{k-2i-1} \binom{\frac{k-1}{2}}{i} - \binom{\frac{k-1}{2}+i}{2i} \right\}.$$

Similarly, $U_{k^n}^k$ can be expanded by means of the binomial theorem

$$\begin{aligned}U_{k^n}^k &= \left\{ \frac{\alpha^{k^n} - \beta^{k^n}}{\alpha - \beta} \right\}^k = \frac{1}{\Delta^{k/2}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \alpha^{(k-j)(k^n)} \beta^{jk^n} \\ &= \sum_{0 \le j \le k/2} (-1)^j \binom{k}{j} \frac{\alpha^{(k-j)k^n} \beta^{jk^n} + (-1)^k \alpha^{jk^n} \beta^{(k-j)k^n}}{(1+\chi(k=2j))\Delta^{k/2}}.\end{aligned} \tag{18}$$

When k is an even integer, the last equality can be rewritten as

$$\Delta^{k/2} U_{k^n}^k = V_{k^{n+1}} + (-q^{k^n})^{k/2} \binom{k}{k/2} + \sum_{j=1}^{k/2-1} (-q^{k^n})^j \binom{k}{j} V_{(k-2j)k^n}. \tag{19}$$

Applying (7) to the last $V_{(k-2j)k^n}$ gives rise to the double sum

$$\begin{aligned}\Delta^{k/2}U_{k^n}^k &= V_{k^{n+1}} + (-q^{k^n})^{k/2}\binom{k}{k/2} + 2\sum_{j=1}^{k/2-1}\binom{k}{j}(-q^{k^n})^{k/2} \\ &+ \sum_{j=1}^{k/2-1}\binom{k}{j}\sum_{i=1}^{k/2-j}(-q^{k^n})^{k/2-i}\binom{\frac{k}{2}+i-j}{2i}\frac{k-2j}{\frac{k}{2}+i-j} \\ &= V_{k^{n+1}} + (2^k-2)(-q^{k^n})^{k/2} \\ &+ \sum_{i=1}^{k/2-1}(-q^{k^n})^{k/2-i}\sum_{j=1}^{k/2-i}\binom{k}{j}\binom{\frac{k}{2}+i-j}{2i}\frac{k-2j}{\frac{k}{2}+i-j}.\end{aligned}$$

By carrying out the same procedure as that for Theorem 11, we get the following counterpart of Corollary 8.

Theorem 13 ($n,\ k \in \mathbb{N}$ with k being even)**.**

$$V_{k^{n+1}} = \Delta^{k/2}U_{k^n}^k - \sum_{i=0}^{k/2-1}(-q^{k^n})^{k/2-i}V_{k^n}^{2i}\left\{2^{k-2i}\binom{\frac{k}{2}}{i} - \frac{2k}{k+2i}\binom{\frac{k}{2}+i}{2i}\right\}.$$

When k is an odd integer, the equation (18) reads analogously as

$$U_{k^n}^k = \frac{1}{\Delta^{(k-1)/2}}\left\{U_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}}(-q^{k^n})^j\binom{k}{j}U_{(k-2j)k^n}\right\}. \tag{20}$$

According to (6), we have the equality

$$U_{(k-2j)k^n} = U_{k^n}\sum_{i=0}^{\frac{k-1}{2}-j}(-q^{k^n})^{\frac{k-1}{2}-j-i}\binom{\frac{k-1}{2}-j+i}{2i}V_{k^n}^{2i}$$

which leads to the double sum expression

$$U_{k^n}^k = \frac{1}{\Delta^{(k-1)/2}}\left\{U_{k^{n+1}} + U_{k^n}\sum_{i=0}^{(k-3)/2}(-q^{k^n})^{\frac{k-1}{2}-i}V_{k^n}^{2i}\sum_{j=1}^{\frac{k-1}{2}-i}\binom{k}{j}\binom{\frac{k-1}{2}-j+i}{2i}\right\}.$$

Following the same derivation as that for Theorem 12, we find the following interesting formula, which is substantially different from Proposition 5.

Theorem 14 ($n,\ k \in \mathbb{N}$ with k being odd)**.**

$$U_{k^{n+1}} = \Delta^{(k-1)/2}U_{k^n}^k - U_{k^n}\sum_{i=0}^{(k-3)/2}(-q^{k^n})^{\frac{k-1}{2}-i}V_{k^n}^{2i}\left\{2^{k-2i-1}\binom{\frac{k-1}{2}}{i} - \binom{\frac{k-1}{2}+i}{2i}\right\}.$$

4. Curious Relations for Fibonacci and Lucas Numbers

When $p = -q = 1$, the formulas displayed in Propositions 3–6, Corollaries 7–10, and Theorems 11–14 become polynomial identities for the classical Fibonacci and Lucas numbers. For $k = 3, 4, 5, 6$, the corresponding curious relations including those shown in the introduction are collected as follows, where we suppose $n \in \mathbb{N}$ and exclude the relations corresponding to $k = 2$ due to their trivialness.

$$F_{3^{n+1}} = 5F_{3^n}^3 - 3F_{3^n}, \qquad \text{Corollary 9: } k=3; \qquad (21a)$$
$$F_{3^{n+1}} = F_{3^n}\{L_{3^n}^2 + 1\}, \qquad \text{Proposition 5: } k=3; \qquad (21b)$$
$$L_{3^{n+1}} = L_{3^n}^3 + 3L_{3^n}, \qquad \text{Corollary 10: } k=3; \qquad (21c)$$
$$L_{3^{n+1}} = L_{3^n}\{5F_{3^n}^2 - 1\}, \qquad \text{Proposition 6: } k=3. \qquad (21d)$$

We remark that both recurrence relations (21a) and (21c) of the first order can also be derived from Theorem 14 and Theorem 12, respectively.

$$F_{4^{n+1}} = L_{4^n}\{5F_{4^n}^3 + 2F_{4^n}\}, \qquad \text{Proposition 3: } k=4; \qquad (22a)$$
$$F_{4^{n+1}} = F_{4^n}\{L_{4^n}^3 - 2L_{4^n}\}, \qquad \text{Proposition 4: } k=4; \qquad (22b)$$
$$L_{4^{n+1}} = 25F_{4^n}^4 + 20F_{4^n}^2 + 2, \qquad \text{Corollary 7: } k=4; \qquad (22c)$$
$$L_{4^{n+1}} = L_{4^n}^4 - 4L_{4^n}^2 + 2, \qquad \text{Corollary 8: } k=4; \qquad (22d)$$
$$L_{4^{n+1}} = L_{4^n}^4 - 20F_{4^n}^2 - 14, \qquad \text{Theorem 11: } k=4; \qquad (22e)$$
$$L_{4^{n+1}} = 25F_{4^n}^4 + 4L_{4^n}^2 - 14, \qquad \text{Theorem 13: } k=4. \qquad (22f)$$

$$F_{5^{n+1}} = F_{5^n}\{L_{5^n}^4 + 3L_{5^n}^2 + 1\}, \qquad \text{Proposition 5: } k=5; \qquad (23a)$$
$$F_{5^{n+1}} = 25F_{5^n}^5 - 25F_{5^n}^3 + 5F_{5^n}, \qquad \text{Corollary 9: } k=5; \qquad (23b)$$
$$F_{5^{n+1}} = 25F_{5^n}^5 - 5F_{5^n}L_{5^n}^2 + 15F_{5^n}, \qquad \text{Theorem 14: } k=5; \qquad (23c)$$
$$L_{5^{n+1}} = L_{5^n}\{25F_{5^n}^4 - 15F_{5^n}^2 + 1\}, \qquad \text{Proposition 6: } k=5; \qquad (23d)$$
$$L_{5^{n+1}} = L_{5^n}^5 + 5L_{5^n}^3 + 5L_{5^n}, \qquad \text{Corollary 10: } k=5; \qquad (23e)$$
$$L_{5^{n+1}} = L_{5^n}^5 + 25L_{5^n}F_{5^n}^2 - 15L_{5^n}, \qquad \text{Theorem 12: } k=5. \qquad (23f)$$

$$F_{6^{n+1}} = L_{6^n}\{25F_{6^n}^5 + 20F_{6^n}^3 + 3F_{6^n}\}, \qquad \text{Proposition 3: } k=6; \qquad (24a)$$
$$F_{6^{n+1}} = F_{6^n}\{L_{6^n}^5 - 4L_{6^n}^3 + 3L_{6^n}\}, \qquad \text{Proposition 4: } k=6; \qquad (24b)$$
$$L_{6^{n+1}} = 125F_{6^n}^6 + 150F_{6^n}^4 + 45F_{6^n}^2 + 2, \qquad \text{Corollary 7: } k=6; \qquad (24c)$$
$$L_{6^{n+1}} = L_{6^n}^6 - 6L_{6^n}^4 + 9L_{6^n}^2 - 2, \qquad \text{Corollary 8: } k=6; \qquad (24d)$$
$$L_{6^{n+1}} = L_{6^n}^6 - 150F_{6^n}^4 - 195F_{6^n}^2 - 62, \qquad \text{Theorem 11: } k=6; \qquad (24e)$$
$$L_{6^{n+1}} = 125F_{6^n}^6 + 6L_{6^n}^4 - 39L_{6^n}^2 + 62, \qquad \text{Theorem 13: } k=6. \qquad (24f)$$

References

[1] W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Cambridge University Press, Cambridge, 1935.

[2] L. Carlitz, *A Fibonacci array,* The Fibonacci Quarterly, **1.2** (1963), 17–28.

[3] W. Chu and Q. Yan, *Multiple convolution formulae on bivariate Fibonacci and Lucas polynomials,* Utilitas Mathematica, **74** (2007), 145–153.

[4] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions,* Revised and enlarged edition; D. Reidel Publishing Company, Dordrecht, 1974.

[5] P. Filipponi, *On the Fibonacci numbers whose subscript is a power,* The Fibonacci Quarterly, **34.3** (1996), 271–276.

[6] P. Filipponi, *Problem B-769: The recurrence of* F_{3^n}, The Fibonacci Quarterly, **32.4** (1994), 373; Solution and Comment by D. C. Terr and M. S. Klamkin in ibid **33.5** (1995), 468–469.

[7] E. Kilic and E. Tan, *Second order linear recursions whose subscripts are a power,* Miskolc Math. Notes, **11.2** (2010), 151–161.
[8] E. Kilic and E. Tan, *Some subsequences of the generalized Fibonacci and Lucas sequences,* to appear in *Utilitas Mathematica*.
[9] H. S. Wilf, *Generatingfunctionology, (second edition),* Academic Press Inc., London, 1994.

MSC2010: 11B39, 05A15

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
E-mail address: chu.wenchang@unisalento.it

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
E-mail address: lina3718@163.com

Current address (W. Chu): Dipartimento di Matematica, Università del Salento, Lecce-Arnesano P. O. Box 193, Lecce 73100 Italy