THE ORDER OF APPEARANCE OF INTEGERS AT MOST ONE AWAY FROM FIBONACCI NUMBERS

DIEGO MARQUES

Abstract. Let F_n be the nth Fibonacci number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. For instance, $z(F_m \pm 1) > m = z(F_m)$, for all $m \geq 5$. In this paper, among other things, we provide explicit forms for $z(F_m \pm 1)$ depending on the class of m modulo 4. In particular, $z(F_m \pm 1) \geq \frac{m^2}{2} - 2$, for $m \equiv 0 \pmod 4$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [6] together with its very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications.

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let n be a positive integer number, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n|F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, $z(n) < \infty$ for all $n \geq 1$. The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [8, p. 300]. Indeed, $z(m) < m^2 - 1$, for all $m > 2$ (see [17, Theorem, p. 52]) and in the case of a prime number p, one has the better upper bound $z(p) \leq p + 1$, which is a consequence of the known congruence $F_{p-(p|5)} \equiv 0 \pmod p$, for $p \neq 2, 5$, where $(p|5)$ denotes the Legendre symbol. Also, it is a simple matter to prove that $z(F_m \pm 1) > m = z(F_m)$, for $m \geq 5$. In fact, if $z(F_m + \epsilon) = j_\epsilon$, with $\epsilon \in \{\pm 1\}$, then $F_m + \epsilon$ divides F_{j_ϵ}, and thus $F_{j_\epsilon} = u(F_m + \epsilon)$ with $u \geq 2$. Therefore, the inequality $F_{j_\epsilon} \geq 2F_m + 2\epsilon > F_m$ gives $z(F_m + \epsilon) = j_\epsilon > m = z(F_m)$. In a very recent paper, the author [9] proved that there exist infinitely many natural numbers n that do not belong to the Fibonacci sequence and such that $z(n \pm 1) > z(n)$. After this brief background, several related problems arise, such as:

- How large is $z(F_m \pm 1)$ compared with m?
- Do the other patterns exist infinitely often, such as $z(m-1) < z(m) < z(m+1)$, or reverse both inequalities, or reverse just the second one?

The aim of this paper is to work on these problems. More precisely, our main result is the following.

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Theorem 1.1. *We have*

(i) $z(F_{4m} \pm 1) = 2(4m^2 - 1)$, *if* $m > 1$.
(ii) $2z(F_{4m+1} - 1) = z(F_{4m+1} + 1) = 4m(2m + 1)$, *if* $m \geq 1$.
(iii)
- $z(F_{8m+2} + 1) = 8m(2m + 1)$, *if* $m \geq 1$;
- $z(F_{8m+2} - 1) = 12m(2m + 1)$, *if* $m \geq 1$;
- $z(F_{8m+6} + 1) = 12(m + 1)(2m + 1)$, *if* $m \geq 0$;
- $z(F_{8m+6} - 1) = 8(m + 1)(2m + 1)$, *if* $m \geq 0$;

(iv) $2z(F_{4m+3} - 1) = z(F_{4m+3} + 1) = 4(m + 1)(2m + 1)$, *if* $m \geq 1$.

As a consequence of the previous theorem, we obtain the following corollary.

Corollary 1.2. *We have*

(i) $z(F_{12m}^2 - 1) = 2(36m^2 - 1)$, *if* $m \geq 1$.
(ii) $z(F_{12m+9}^2 - 1) = 4(3m + 2)(6m + 5)$, *if* $m \geq 0$.
(iii)
- $z(F_{24m+12}^2 - 1) \in \{12(3m + 2)(6m + 5), 24(3m + 2)(6m + 5)\}$, *if* $m \geq 0$;
- $z(F_{24m+6}^2 - 1) \in \{12(3m + 1)(6m + 1), 24(3m + 1)(6m + 1)\}$, *if* $m \geq 0$;

(iv) $z(F_{12m+3}^2 - 1) = 4(3m + 1)(6m + 1)$, *if* $m \geq 0$
(v) *If* $3 \nmid m$, *then* $z((F_{4m}^2 - 1)/2) = 2(4m^2 - 1)$;
(vi) *If* $3 \nmid m + 1$, *then* $z((F_{4m+1}^2 - 1)/2) = 4m(2m + 1)$;
(vii) *If* $3 \nmid m + 1$, *then* $z((F_{8m+2}^2 - 1)/2) \in \{12m(2m + 1), 24m(2m + 1)\}$;
(viii) *If* $3 \nmid m$, *then* $z((F_{8m+6}^2 - 1)/2) \in \{12(m + 1)(2m + 1), 24(m + 1)(2m + 1)\}$;
(ix) *If* $3 \nmid m$, *then* $z((F_{4m+3}^2 - 1)/2) = 4(m + 1)(2m + 1)$.

We remark that the problem of finding some solution for the Diophantine equation $z(n) = z(n+1)$ remains open, however Theorem 1.1 (i) and Corollary 1.2 (v) provides infinitely many solutions for $z(n) = z(n + 2) = z(n(n + 2)/2)$, namely $n = F_{4m} - 1$, for all $m > 1$, with $3 \nmid m$.

The next result concerns the possible behavior of the order of appearance in three consecutive integers.

Theorem 1.3. *For* $t \geq 0$, *set* $N_t = F_{12t+7} + 1$ *and* $M_t = F_{12t+3} - 1$. *Then*

(i) $z(N_t - 1) < z(N_t) > z(N_t + 1)$;
(ii) $z(M_t - 1) > z(M_t) > z(M_t + 1)$, *if* $6|t$.

Note that we did not prove that $z(n - 1) < z(n) < z(n + 1)$ holds infinitely often.

We organize this paper as follows. In Section 2, we will recall some useful properties of Fibonacci and Lucas numbers. Section 3 is devoted to the proof of theorems. In the last section, we prove a result concerning the order of appearance of the product of Fibonacci by Lucas numbers and we make two related conjectures.

2. Auxiliary Results

Before proceeding further, some considerations will be needed for the convenience of the reader.

We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$. First, we recall some classical and helpful facts which will be essential ingredients to prove Theorems 1.1 and 1.3.

Lemma 2.1. *We have*

(a) $F_n|F_m$ *if and only if* $n|m$.

(b) $L_n | F_m$ *if and only if* $n|m$ *and* m/n *is even.*
(c) $L_n | L_m$ *if and only if* $n|m$ *and* m/n *is odd.*
(d) $F_{2n} = F_n L_n$.
(e) *If* $d = \gcd(m,n)$, *then*

$$\gcd(F_m, L_n) = \begin{cases} L_d, & \text{if } m/d \text{ is even and } n/d \text{ is odd;} \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

(f) $2|F_m$ *if and only if* $3|m$, *and* $3|F_m$ *if and only if* $4|m$.
(g) $2|L_m$ *if and only if* $3|m$.
(h) $F_{3n} = 5F_n^3 + 3(-1)^n F_n$.
(i) $3F_{4n} | F_{12n}$.
(j) *(d'Ocagne's identity)* $(-1)^n F_{m-n} = F_m F_{n+1} - F_n F_{m+1}$.

Most of the previous items can be proved by using the well-known Binet's formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n, \text{ for } n \geq 0,$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Note also that there are some implications among these items, such as (a) $\Rightarrow$ (f), (e) $\Rightarrow$ (b), (c) $\Rightarrow$ (g), and (h) $\Rightarrow$ (i). We refer the reader to [1, 5, 6, 13] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

Lemma 2.2. *We have*

(a) *If* $F_n | m$, *then* $n | z(m)$.
(b) *If* $L_n | m$, *then* $2n | z(m)$.
(c) *If* $n | F_m$, *then* $z(n) | m$.
(d) *Let* a *and* b *be positive integers. If* $az(m) = bz(n)$, *then*

$$\max\{z(m), z(n)\} | z(\text{lcm}(m,n)) | az(m).$$

In particular, if $z(n)|z(m)$, *then* $z(\text{lcm}(m,n)) = z(m)$.

Proof. For (a) and (b), since $F_n | m | F_{z(m)}$, by Lemma 2.1 (a), we get $n|z(m)$. Also, $L_n | m | F_{z(m)}$ and by Lemma 2.1 (b), we have that $z(m)/n$ is even. In particular, $2n|z(m)$. In order to prove (c), we write $m = z(n)q + r$, where q and r are integers, with $0 \leq r < z(n)$. So, by Lemma 2.1 (j), we obtain

$$(-1)^{z(n)q} F_r = F_m F_{z(n)+1} - F_{z(n)} F_{m+1}.$$

Since n divides both F_m and $F_{z(n)}$, then it also divides F_r implying $r = 0$ (keep in mind the range of r). Thus $z(n)|m$. For the last item, one has $n|F_{z(n)}|F_{bz(n)}$, so $n|F_{az(m)}$. On the other hand, $m|F_{z(m)}|F_{az(m)}$ leading to $\text{lcm}(m,n)|F_{az(m)}$ and then $z(\text{lcm}(m,n))|az(m)$, by the previous item. Also, we use that $\max\{m,n\}|\text{lcm}(m,n)|F_{z(lcm(m,n))}$, together with (c), for yielding $\max\{z(m), z(n)\}|z(\text{lcm}(m,n))$. □

The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. The p-adic order of a Fibonacci number was completely characterized, see [4, 7, 15, 16]. For instance, from the main theorem of Lengyel [7], we extract the following result.

Lemma 2.3. *For* $n \geq 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \textit{if } n \equiv 1,2 \pmod 3; \\ 1, & \textit{if } n \equiv 3 \pmod 6; \\ 3, & \textit{if } n \equiv 6 \pmod{12}; \\ \nu_2(n)+2, & \textit{if } n \equiv 0 \pmod{12}. \end{cases}$$

$$\nu_3(F_n) = \begin{cases} \nu_3(n)+1, & \textit{if } n \equiv 0 \pmod 4; \\ 0, & \textit{otherwise.} \end{cases}$$

A proof of a general statement can be found in [7, p. 236-237].

The equation $F_n+1=y^2$ and more generally $F_n \pm 1 = y^\ell$ with integer y and $\ell \geq 2$ have been solved in [14] and [3], respectively. The solution for the last equation makes appeal to Fibonacci and Lucas numbers with negative indices which are defined as follows: let $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example, $F_{-1} = 1, F_{-2} = -1$, and so on. In general, $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$, for $n > 0$. Bugeaud et al. [3, Section 5] used these numbers to give factorizations for $F_m \pm 1$. More recently, the author [10, 11, 12] used this method together with the *Primitive Divisor Theorem* (see [2]) to work on the Diophantine equations $F_1 \cdots F_n + 1 = F_m^t$ and $\left[{m \atop k}\right]_F = F_n \pm 1$, where

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m F_{m-1} \cdots F_{m-k+1}}{F_1 \cdots F_k}$$

are the known *Fibonomial coefficients*. For the sake of completeness, let us sketch the Bugeaud et al method.

Since the Binet's formulas remain valid for Fibonacci and Lucas numbers with negative indices, one can deduce the following result.

Lemma 2.4. *For any integers a, b, we have*

$$F_a L_b = F_{a+b} + (-1)^b F_{a-b}.$$

Proof. The identity $\alpha = (-\beta)^{-1}$ leads to

$$F_a L_b = \frac{\alpha^a - \beta^a}{\alpha - \beta}(\alpha^b + \beta^b) = F_{a+b} + \frac{\alpha^a\beta^b - \beta^a\alpha^b}{\alpha - \beta} = F_{a+b} + (-1)^b F_{a-b}.$$

□

Lemma 2.4 immediately gives the following factorizations for $F_m \pm 1$, depending on the class of m modulo 4: $F_m \pm 1 = F_a L_b$, where $2a, 2b \in \{m \pm 1, m \pm 2\}$.

Before proceeding further, we shall assume Theorem 1.1 in order to prove its corollary.

Proof of Corollary 1.2. (i) By Theorem 1.1, $z(n) = z(n+2) = 2(36m^2-1)$, where $n = F_{12m} - 1$. The result follows from Lemma 2.2 (c), because

$$z(F_{12m}^2 - 1) = z(n(n+2)) = z(\mathrm{lcm}(n, n+2)) = z(n),$$

where we use the fact that $\mathrm{lcm}(n, n+2) = n(n+2)$, since n is odd.

The other cases can be handled in much the same way by using Lemma 2.2 (c) together with relations

$$z(F_{4m+\delta}+1) = 2z(F_{4m+\delta}-1),\ 3z(F_{8m+2}+1) = 2z(F_{8m+2}-1) \text{ and}$$
$$2z(F_{8m+6}+1) = 3z(F_{8m+6}-1),$$

where $\delta \in \{1, 3\}$.

□

Now, we are ready to deal with the proof of the theorems.

3. The Proof of Theorems

3.1. **The Proof of Theorem 1.1.** *(i)* Taking $a = 2m + \epsilon_1$ and $b = 2m + \epsilon_2$ in Lemma 2.4, where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ are distinct, we get $F_{2m+\epsilon_1} L_{2m+\epsilon_2} = F_{4m} \pm 1$. Note that $F_{2(2m+\epsilon_1)(2m+\epsilon_2)} = F_{(2m+\epsilon_1)(2m+\epsilon_2)} L_{(2m+\epsilon_1)(2m+\epsilon_2)}$ (by Lemma 2.1 (d)). Since $F_{2m+\epsilon_1} | F_{(2m+\epsilon_1)(2m+\epsilon_2)}$ and $L_{2m+\epsilon_2} | L_{(2m+\epsilon_1)(2m+\epsilon_2)}$, by Lemma 2.1 (a) and (c), we have

$$F_{4m} \pm 1 = F_{2m+\epsilon_1} L_{2m+\epsilon_2} | F_{2(4m^2-1)}$$

which yields $z(F_{4m} \pm 1) \le 2(4m^2 - 1)$.

On the other hand, both $F_{2m+\epsilon_1}$ and $L_{2m+\epsilon_2}$ divide $F_{4m} \pm 1$ which, by Lemma 2.2, ensures that both $2m + \epsilon_1$ and $2(2m + \epsilon_2)$ divide $z(F_{4m} \pm 1)$. Since $\gcd(2m + \epsilon_1, 2(2m + \epsilon_2)) = 1$, then $2(4m^2 - 1) | z(F_{4m} \pm 1)$ and hence $z(F_{4m} \pm 1) \ge 2(4m^2 - 1)$. Thus, we have the desired equality.

(ii) and (iv) In Lemma 2.4, take $a = 2m + \epsilon_1,\ b = 2m + \epsilon_2$, where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ are distinct, and $\delta \in \{0, 2\}$. Then

$$F_{4m+1+\delta} + 1 = F_{2m+1} L_{2m+\delta} \quad \text{and} \quad F_{4m+1+\delta} - 1 = F_{2m+\delta} L_{2m+1}.$$

Observe that

$$F_{4m+1+\delta} + 1 = F_{2m+1} L_{2m+\delta} | F_{2(2m+1)(2m+\delta)} = F_{(2m+1)(2m+\delta)} L_{(2m+1)(2m+\delta)},$$

because $L_{2m+\delta} | L_{(2m+1)(2m+\delta)}$ (Lemma 2.1 (c)). Also, $\gcd(2m + 1, 2(2m + \delta)) = 1$ and so the proof of $z(F_{4m+1+\delta} + 1) = 2(2m + 1)(2m + \delta)$ is very similar to the previous item. However, the (-) case is more interesting, since $L_{2m+1} \nmid L_{(2m+1)(2m+\delta)}$, because $(2m+1)(2m+\delta)/(2m+1)$ is even. Despite that, $L_{2m+1} | F_{(2m+1)(2m+\delta)}$ (Lemma 2.1 (d)). Also, by Lemma 2.1 (e), $\gcd(F_{2m+\delta}, L_{2m+1}) = 1$ and therefore

$$F_{4m+1+\delta} - 1 = F_{2m+\delta} L_{2m+1} | F_{(2m+1)(2m+\delta)},$$

yielding $z(F_{4m+1+\delta} - 1) \le (2m + 1)(2m + \delta)$. On the other hand, by the factorization of $F_{4m+1+\delta} - 1$, we get by Lemma 2.2 that both $2m + 1$ and $2m + \delta$ divide $z(F_{4m+1+\delta} - 1)$. Since $\gcd(2m + 1, 2m + \delta) = 1$, we have that $(2m + 1)(2m + \delta) | z(F_{4m+1+\delta} - 1)$ and then $z(F_{4m+1+\delta} - 1) \ge (2m + 1)(2m + \delta)$. Summarizing,

- Case $\delta = 0$: $z(F_{4m+1} - 1) = 2m(2m + 1) = z(F_{4m+1} + 1)/2$;
- Case $\delta = 2$: $z(F_{4m+3} - 1) = 2(m + 1)(2m + 1) = z(F_{4m+3} + 1)/2$.

(iii) The cases $F_{8m+2} + 1$ and $F_{8m+6} - 1$. Set $\delta \in \{0, 4\}$, by Lemma 2.4, we have

$$F_{8m+2+\delta} + (-1)^{\delta/4} = F_{4m+2} L_{4m+\delta}.$$

Note that both F_{4m+2} and $L_{4m+\delta}$ divide $F_{2(2m+1)(4m+\delta)}$. Since $d = \gcd(4m + 2, 4m + \delta) = 2$, $(4m + 2)/d = 2m + 1$ and $(4m + \delta)/d = 2m + \delta/2$ (even), Lemma 2.1 (e) implies $\gcd(F_{4m+2}, L_{4m+\delta}) = 1$ or 2. However, if F_{4m+2} and $L_{4m+\delta}$ are even numbers, then 3 divides both $4m + 2$ and $4m + \delta$ (Lemma 2.1 (f) and (g)) leading to an absurdity as $3 | (\delta - 2) \in \{\pm 2\}$. Hence $\gcd(F_{4m+2}, L_{4m+\delta}) = 1$ and so

$$F_{8m+2+\delta} + (-1)^{\delta/4} | F_{2(2m+1)(4m+\delta)}.$$

Thus $z(F_{8m+2+\delta} + (-1)^{\delta/4}) \le 2(2m + 1)(4m + \delta)$. For the opposite inequality, we use that $F_{2m+1} | F_{4m+2} | F_{8m+2+\delta} + (-1)^{\delta/4}$, leading to $(2m + 1) | z(F_{8m+2+\delta} + (-1)^{\delta/4})$. Also, $L_{4m+\delta}$ divides $F_{8m+2+\delta} + (-1)^{\delta/4}$ and so $2(4m + \delta) | z(F_{8m+2+\delta} + (-1)^{\delta/4})$. Again, we use the fact that $\gcd(2m + 1, 2(4m + \delta)) = 1$ for yielding

$$z(F_{8m+2+\delta} + (-1)^{\delta/4}) \ge 2(2m + 1)(4m + \delta).$$

- Case $\delta = 0$: $z(F_{8m+2}+1) = 8m(2m+1)$;
- Case $\delta = 4$: $z(F_{8m+6}-1) = 8(m+1)(2m+1)$.

The cases $F_{8m+2}-1$ *and* $F_{8m+6}+1$. We have

$$F_{8m+2+\delta} + (-1)^{(\delta-4)/4} = F_{4m+\delta}L_{4m+2}.$$

Note that, both $F_{4m+\delta}$ and L_{4m+2} divide $F_{(4m+\delta)(2m+1)}$ and since

$$\gcd(F_{4m+\delta}, L_{4m+2}) = L_{\gcd(4m+\delta,4m+2)} = L_2 = 3,$$

then

$$F_{4m+\delta}L_{4m+2} | 3F_{(4m+\delta)(2m+1)} | F_{3(4m+\delta)(2m+1)},$$

where we used the fact that $(4m+\delta)(2m+1)$ is a multiple of 4 together with Lemma 2.1 (i). Hence, by Lemma 2.2 (d),

$$z(F_{8m+2+\delta} + (-1)^{(\delta-4)/4}) | 3(4m+\delta)(2m+1).$$

As before, the factorization of $F_{8m+2+\delta} + (-1)^{(\delta-4)/4}$ gives $(4m+\delta)(2m+1) | z(F_{8m+2+\delta} + (-1)^{(\delta-4)/4})$. Hence, we conclude that $z(F_{8m+2+\delta} + (-1)^{(\delta-4)/4})$ belongs to

$$\{(4m+\delta)(2m+1), 3(4m+\delta)(2m+1)\}, \quad \text{for all} \quad m \geq 1.$$

Now, it suffices to prove that $F_{8m+2+\delta} + (-1)^{(\delta-4)/4} \nmid F_{(4m+\delta)(2m+1)}$. Towards a contradiction, suppose that, on the contrary, there is an integer ℓ such that

$$F_{(4m+\delta)(2m+1)} = \ell(F_{8m+2+\delta} + (-1)^{(\delta-4)/4}).$$

By factoring the right-hand side above, together with Lemma 2.1 (d), we get

$$F_{(4m+\delta)(2m+1)}F_{4m+2} = \ell F_{4m+\delta}F_{8m+4},$$

which yields $\nu_3(F_{(4m+\delta)(2m+1)}) \geq \nu_3(F_{4m+\delta}F_{8m+4})$. However, Lemma 2.3 gives

$$\nu_3(F_{(4m+\delta)(2m+1)}) = \nu_3((4m+\delta)(2m+1)) + 1,$$

while

$$\nu_3(F_{4m+\delta}F_{8m+4}) = \nu_3(F_{4m+\delta}) + \nu_3(F_{8m+4}) = \nu_3((4m+\delta)(2m+1)) + 2,$$

which is a contradiction. This finishes the proof.

– Case $\delta = 0$: $z(F_{8m+2}-1) = 12m(2m+1)$;
– Case $\delta = 4$: $z(F_{8m+6}+1) = 12(m+1)(2m+1)$.

□

3.2. **The Proof of Theorem 1.3.** *(i)* Take $a = 2m+3$ and $b = 2m$ in Lemma 2.4, so we get

$$F_{2m+3}L_{2m} = F_{4m+3} + 2.$$

We apply the same method as in the proof of Theorem 1.1 to get the estimate $z(F_{4m+3}+2) \leq 4m(2m+3)$. Moreover, both F_{2m+3} and L_{2m} divide $F_{4m+3}+2$ which implies, by Lemma 2.2 (a) and (b), that both $2m+3$ and $4m$ divide $z(F_{4m+3}+2)$. Now, if $3 \nmid m$, then $\gcd(2m+3, 4m) = 1$ and thus $z(F_{4m+3}+2) \geq 4m(2m+3)$. Hence, $z(F_{4m+3}+2) = 4m(2m+3)$ if $3 \nmid m$. In particular, if $m = 3t+1$, then $z(N_t+1) = 4(3t+1)(6t+5)$.

On the other hand, taking $m = 3t+1$ in Theorem 1.1 (iv), we obtain

$$z(N_t) = 4(3t+2)(6t+3).$$

Now, a straight calculation gives the desired result. In fact,

$$z(N_t) - z(N_t+1) = 4, \quad \text{for all} \quad t \geq 0.$$

(ii) Taking $a = 6t$ and $b = 6t+3$ in Lemma 2.4, we obtain
$$M_t - 1 = F_{12t+3} - 2 = F_{6t}L_{6t+3}.$$
By Lemma 2.2 (c), it follows that both $6t$ and $6t+3$ divide $z(M_t-1)$. Since $\gcd(6t, 6t+3) = 3$, then $2t(6t+3)|z(M_t-1)$. Thus,
$$z(M_t-1) \in \{2t(6t+3), 4t(6t+3), 6t(6t+3), 8t(6t+3), \ldots\}.$$
We claim that $(M_t - 1) \nmid F_{6t(6t+3)}$. To derive a contradiction, we suppose that $F_{6t(6t+3)} = \ell(M_t-1)$, for some integer ℓ. This equality becomes
$$F_{6t(6t+3)}F_{6t+3} = \ell F_{6t}F_{12t+6}.$$
So, $\nu_2(F_{6t(6t+3)}F_{6t+3}) \geq \nu_2(F_{6t}F_{12t+6})$. Now, since t is even, Lemma 2.3 gives
$$\nu_2(F_{6t(6t+3)}F_{6t+3}) = \nu_2(6t) + \nu_2(6t+3) + 3 = \nu_2(t) + 4,$$
while
$$\nu_2(F_{6t}F_{12t+6}) = \nu_2(6t) + 5 = \nu_2(t) + 6.$$
Therefore, $(M_t - 1) \nmid F_{6t(6t+3)}$ yielding
$$z(M_t-1) \in \{4t(6t+3), 8t(6t+3), 10t(6t+3), 12t(6t+3), \ldots\}.$$
The fact that $(M_t - 1) \nmid F_{4t(6t+3)}$ is proved similarly. Indeed
$$\nu_2(M_t-1) = \nu_2(F_{6t(6t+3)}) + 1, \quad \text{if } 3|t.$$
We then conclude that
$$z(M_t-1) \geq 8t(6t+3) > 2(3t+1)(6t+1) = z(M_t) > z(M_t+1) = 12t+3,$$
where we used the Theorem 1.1 (iv) for $m = 3t$. □

4. Further Comments and Some Conjectures

It is a simple matter to deduce from the Primitive Divisor Theorem that 1, 2, and 3 are the only integers which are both Fibonacci and Lucas numbers. Thus, what about the order of appearance of terms of Lucas sequence? The answer is a consequence of the next result.

Proposition 4.1. *If m and n are positive integers, with m odd, $n > 1$ and $\gcd(m,n) = 1$, then $z(F_mL_n) = 2mn$. In particular, $z(L_n) = 2n$, for all $n > 1$.*

Proof. Since m is odd, then Lemma 2.1 (c) implies that $L_n|L_{mn}$. Thus, $F_mL_n|F_{mn}L_{mn} = F_{2mn}$ yielding $z(F_mL_n)|2mn$. However, Lemma 2.2 immediately gives that $m|z(F_mL_n)$ and $2n|z(F_mL_n)$. We use that $\gcd(m, 2n) = 1$, to get $2mn|z(F_mL_n)|2mn$ which gives the desired equality. □

A natural problem is to find closed forms for $z(L_m \pm 1)$. For that we can use a "Lucas" version of Lemma 2.4. For all integers a and b,
$$L_aL_b = L_{a+b} + (-1)^bL_{a-b}.$$
Thus, we get factorizations for $L_m \pm 1$. But, unlike the Fibonacci case, we cannot factor, for instance, $L_{4m} \pm 1$, $L_{4m+1} - 1$, and $L_{4m+3} + 1$. The useful fact for factoring $F_{4m} \pm 1$ is that $F_2 = 1$ which does not happen in the Lucas sequence. In fact, $F_n = 1$, for $n = -1, 1, 2$, while $L_n = 1$ only for $n = 1$. However, when the factorization is possible we can deduce, similarly to Theorem 1.1, that
$$z(L_{4m+1}+1) = 4m(2m+1) \quad \text{and} \quad z(L_{4m+3}-1) = 4(m+1)(2m+1).$$

We finish by stating two conjecture which will be left for the reader. The first one concerns the order of appearance of $L_m + j$, for $j = -1, 0, 1, 2$.

Conjecture 4.2. *We have*

(i) $z(L_{2m+1} - 1) > z(L_{2m+1}) < z(L_{2m+1} + 1)$, *if* $m \geq 2$;
(ii) $z(L_{4m}) > z(L_{4m} + 1) > z(L_{4m} + 2)$, *if* $m \geq 1$.

We point out that we were not able to prove the existence of infinitely many three consecutive integers with increasing order of appearance. The below conjecture is related to this case.

Conjecture 4.3. *We have*

(i) $z(L_{4m+2}) < z(L_{4m+2} + 1) < z(L_{4m+2} + 2)$, *if* $m \geq 0$;
(ii) $z(F_{8m+2}) < z(F_{8m+2} + 1) < z(F_{8m+2} + 2)$, *if* $m \geq 1$.

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Departamento de Matemática, Universidade de Brasília, Brasília, DF, 70910-900, Brazil
E-mail address: diego@mat.unb.br