ON SOME NEW SUMS OF FIBONOMIAL COEFFICIENTS

DIEGO MARQUES AND PAVEL TROJOVSKÝ

ABSTRACT. Let F_n be the nth Fibonacci number. The Fibonomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_F$ are defined for $n \geq k > 0$ as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_F = 0$ for $n < k$. In this paper, we shall provide some interesting sums among Fibonomial coefficients. In particular, we prove that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F F_{n+4m+2-j} = 0,$$

holds for all non-negative integers m and n.

1. Introduction

In 1915, Fontené published a one-page note [2] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{a_n\}$ of real or complex numbers. Thus the generalized binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{a_n a_{n-1} \cdots a_{n-k+1}}{a_1 a_2 \cdots a_k}.$$

Setting $a_n = n$ we recover the ordinary binomial coefficients, while $a_n = q^n - 1$ we obtain the q-binomial coefficients studied by Gauss, Euler, and Cauchy and which were shortly called q-Gaussian coefficients (Gauss q-binomial coefficients). The sequence $\{a_n\}$ is essentially arbitrary but we do require that $a_n \neq 0$ for $n \geq 1$.

Since 1964 there has been an accelerated interest in *Fibonomial coefficients*, which correspond to the choice $a_n = F_n$, where F_n is the nth Fibonacci number. During the last decades several identities among these numbers have been found. Gould [3] derived the relation

$$\sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_F = \begin{bmatrix} n \\ k \end{bmatrix}_F.$$

Lind [7], using a result from a paper of Jarden and Motzkin [4], obtained the identity

$$\sum_{j=0}^{k} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k \\ j \end{bmatrix}_F F_{n-j}^{k-1} = 0,$$

where n, k are any positive integers such that $n \geq k$, and further he found the formula

The first author was supported in part by FEMAT-Brazil and CNPq-Brazil.

$$\sum_{j=0}^{k+1}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}k+1\\ j\end{bmatrix}_F\begin{bmatrix}n-j\\ k\end{bmatrix}_F=0.$$

Seibert and Trojovský [8] included identities

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(m+i)}\begin{bmatrix}m\\ i\end{bmatrix}_F=0,$$

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i+1)}\frac{F_{(k-i)(k-2l)}}{F_{k-2l}}\begin{bmatrix}k+1\\ i\end{bmatrix}_F=0$$

and

$$\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2l+i+(-1)^k)}L_{(i+n)(k-2l)}\begin{bmatrix}k+1\\ i\end{bmatrix}_F=0$$

for any positive integers m, k, n and l, with m odd, $l<(k-1)/2$ and $m>k$. Here, L_n denotes the nth Lucas number. Kiliç et al. [5] proved the following formula

$$\sum_{j=0}^{m-1}(-1)^{\frac{j}{2}(j+3)}\begin{bmatrix}(m+1)k+m\\ j\end{bmatrix}_F\begin{bmatrix}(m+1)k+m-j-1\\ m-j-1\end{bmatrix}_F F_{n+k+m-j}^{m+1}$$
$$+(-1)^{\frac{m}{2}(m+3)}F_{n-mk}^{m+1}=F_{(m+1)(n+\frac{m}{2})}\prod_{j=1}^{m}F_{(m+1)k+j},$$

where m, n, and k are any integers. We refer the reader to [6] for related identities involving generalized Fibonomial coefficients.

In 2007, as Problem B-1040 of the problem section of *The Fibonacci Quarterly*, Bruckman [1] proposed the problem of finding a proof of identity

$$\sum_{j=0}^{4m}(-1)^{\frac{k}{2}(k+1)}\begin{bmatrix}4m\\ k\end{bmatrix}_F F_k=0.$$

The aim of this paper is to provide some identities involving sums of Fibonomial coefficients. In particular, we shall give a generalization of the previous formula. More precisely, our main results are the following.

Theorem 1.1. *Let m, n be any non-negative integer. Then*

$$\sum_{j=0}^{4m+2}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j}=\frac{1}{2}F_{2m+n+1}\sum_{j=0}^{4m+2}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F L_{2m+1-j}$$

and

$$\sum_{j=0}^{4m}(-1)^{\frac{j}{2}(j-1)}\begin{bmatrix}4m\\ j\end{bmatrix}_F F_{n+4m-j}=\frac{1}{2}F_{2m+n}\sum_{j=0}^{4m}(-1)^{\frac{j}{2}(j-1)}\begin{bmatrix}4m\\ j\end{bmatrix}_F L_{2m-j}.$$

Theorem 1.2. *Let m, n be any non-negative integers. Then*

$$\sum_{j=0}^{4m+2}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j}=0. \tag{1.1}$$

and

$$\sum_{j=0}^{4m} (-1)^{\frac{j}{2}(j-1)} \begin{bmatrix} 4m \\ j \end{bmatrix}_F F_{n+4m-j} = 0. \tag{1.2}$$

2. A Key Auxiliary Result

Before proceeding further, we shall prove a fact which will be an essential ingredient in the proof of Theorem 1.2.

Lemma 2.1. *Let m and k be any non-negative integers. Then*

$$\sum_{j=4k+3}^{4k+6} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j} = -\begin{bmatrix} 4m+2 \\ 4k+7 \end{bmatrix}_F \frac{F_{4k+7}}{F_{2m+1}} + \begin{bmatrix} 4m+2 \\ 4k+3 \end{bmatrix}_F \frac{F_{4k+3}}{F_{2m+1}}. \tag{2.1}$$

Proof. Using clear identities

$$\begin{bmatrix} 4m+2 \\ 4k+3+i \end{bmatrix}_F = \begin{bmatrix} 4m+2 \\ 4k+3 \end{bmatrix}_F \prod_{j=0}^{i-1} \frac{F_{4m-4k-1-j}}{F_{4k+4+j}}, \quad i = 1, 2, 3, 4$$

we can rewrite formula (2.1) as the identity

$$\begin{aligned} & F_{2m+1}L_{2m-4k-2}F_{4k+6}F_{4k+5}F_{4k+4} \\ & \quad + F_{4m-4k-1}F_{2m+1}L_{2m-4k-3}F_{4k+6}F_{4k+5} \\ & \quad - F_{4m-4k-1}F_{4m-4k-2}F_{2m+1}L_{2m-4k-4}F_{4k+6} \\ & \quad - F_{4m-4k-1}F_{4m-4k-2}F_{4m-4k-3}F_{2m+1}L_{2m-4k-5} \\ = & \; - F_{4m-4k-1}F_{4m-4k-2}F_{4m-4k-3}F_{4m-4k-4} + F_{4k+6}F_{4k+5}F_{4k+4}F_{4k+3,} \end{aligned}$$

which can be simplified by the identity $F_{n+h}L_{n+k} - F_nL_{n+h+k} = (-1)^n F_h L_k$ (see [9, 19b]), and well-known identity $F_nL_n = F_{2n}$ to the form

$$F_{4k+6}F_{4k+5}F_{4m-4k-1}F_{4m-4k-2} - F_{4m-4k-1}F_{4m-4k-2}F_{4k+6}F_{4k+5} = 0.$$

□

Lemma 2.2. *Let m be any non-negative integer. Then*

$$-2\begin{bmatrix} 4m+2 \\ 3 \end{bmatrix}_F + \begin{bmatrix} 4m+2 \\ 2 \end{bmatrix}_F F_{2m+1}L_{2m-1} + \begin{bmatrix} 4m+2 \\ 1 \end{bmatrix}_F F_{2m+1}L_{2m} = F_{4m+2}.$$

Proof. After overwriting the Fibonomial coefficients using their definition, we get

$$-F_{4m+2}F_{4m+1}F_{4m} + F_{4m+2}F_{4m+1}F_{2m+1}L_{2m-1} + F_{4m+2}F_{2m+1}L_{2m} = F_{4m+2}.$$

On dividing through by F_{4m+2}, a straight calculation gives

$$F_{4m+1}F_{2m+1}L_{2m-1} + F_{2m+1}L_{2m} = F_{4m+1}F_{4m} + 1.$$

Now we use the formulas $F_{4m+1}F_{4m} + 1 = F_{4m-1}F_{4m+2}$, $F_{4m+1} - 1 = F_{2m}L_{2m+1}$, $F_{4m-1} - 1 = F_{2m}L_{2m-1}$ (they are special cases of identities (20a) and (15b) in [9]), to obtain the clear equality $F_{2m}L_{2m+1}L_{2m-1} = F_{2m}L_{2m+1}L_{2m-1}$. □

Lemma 2.3. *Let m and n be any non-negative integers. Then*

$$\sum_{j=0}^{4n+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j} = -\begin{bmatrix} 4m+2 \\ 4n+3 \end{bmatrix}_F \frac{F_{4n+3}}{F_{2m+1}}$$

and

$$\sum_{j=0}^{4n} (-1)^{\frac{j}{2}(j-1)} \begin{bmatrix} 4m \\ j \end{bmatrix}_F L_{2m-j} = \begin{bmatrix} 4m \\ 4n+1 \end{bmatrix}_F \frac{F_{4n+1}}{F_{2m}}.$$

Proof. We shall prove the first identity, because the proofs of both identities are very similar. For that, we use induction on n. For $n = 0$ the assertion is implied by Lemma 2.2. Let us consider that the identity holds for $n = k$ and prove it for $n = k + 1$. The left-hand side can be written as

$$\begin{aligned} \sum_{j=0}^{4k+6} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j} &= \sum_{j=0}^{4k+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j} \\ &+ \sum_{j=4k+3}^{4k+6} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j} \end{aligned}$$

and the identity follows from Lemma 2.1. □

Now, we are ready to deal with the proof of the theorems.

3. The Proof of Theorems

3.1. **The proof of Theorem 1.1.** Again, we shall prove only the first identity, since the proof of the second one can be handled in much the same way (we use the identity $F_{n+4m-j} + (-1)^j F_{n+j} = F_{2m+n} L_{2m-j}$).

$$\begin{aligned} &2 \sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F F_{n+4m+2-j} \\ &= \sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F F_{n+4m+2-j} + \sum_{k=0}^{4m+2} (-1)^{\frac{4m+2-k}{2}(4m+2-k+1)} \begin{bmatrix} 4m+2 \\ 4m+2-k \end{bmatrix}_F F_{n+k} \\ &= \sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F F_{n+4m+2-j} - \sum_{k=0}^{4m+2} (-1)^k (-1)^{\frac{k}{2}(k+1)} \begin{bmatrix} 4m+2 \\ k \end{bmatrix}_F F_{n+k} \\ &= \sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F \left(F_{n+4m+2-j} - (-1)^j F_{n+j} \right) \\ &= F_{2m+n+1} \sum_{j=0}^{4m+2} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{2m+1-j}, \end{aligned}$$

where we use the identity $F_{n+4m+2-j} - (-1)^j F_{n+j} = F_{2m+n+1} L_{2m+1-j}$, which follows from the identity $F_{a+b} - (-1)^b F_{a-b} = F_b L_a$ (see [9, 15b],). □

3.2. **The proof of Theorem 1.2.** Identity (1.1) follows from the first formula in Theorem 1.1 and Lemma 2.3 (with $m = n$). Here we have used the fact that $\left[{4m+2 \atop 4m+3}\right]_F = 0$. Similarly, identity (1.2) follows from the second formula in Theorem 1.1 and Lemma 2.3. □

4. Further Comments, a Conjecture and Its Consequence

In this section, we shall discuss several sums related to identities (1.1) and (1.2).

Theorem 4.1. *The following formulas are equivalent*

(i) $\displaystyle\sum_{j=0}^{4m}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} = 0.$

(ii) $\displaystyle F_n = \sum_{j=0}^{2m}(-1)^{\frac{j(j+1)}{2}}\left(\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} - \begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_{n+j+1}\right).$

(iii) $\displaystyle 2 = \sum_{j=0}^{2m}(-1)^{\frac{j(j+1)}{2}}\left(\begin{bmatrix}4m+2\\ j\end{bmatrix}_F L_{4m+2-j} - \begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F L_{j+1}\right).$

Proof. We rewrite (i) as

$$F_n = \sum_{j=0}^{2m}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} + \sum_{j=2m+1}^{4m+1}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j}.$$

By taking the change of indexes $j = 4m - j + 1$ in the second sum above, we get

$$\begin{aligned} F_n = &\sum_{j=0}^{2m}(-1)^{\frac{j(j+1)}{2}}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j}\\ &+\sum_{j=0}^{2m}(-1)^{\frac{(4m-j+1)(4m-j+2)}{2}}\begin{bmatrix}4m+2\\ 4m-j+1\end{bmatrix}_F F_{n+j+1}. \end{aligned}$$

Since $\left[{4m+2 \atop 4m-j+1}\right]_F = \left[{4m+2 \atop j+1}\right]_F$ and $\frac{(4m-j+1)(4m-j+2)}{2} \equiv -\frac{j(j+1)}{2} \pmod 2$, we obtain

$$F_n = \sum_{j=0}^{2m}(-1)^{\frac{j(j+1)}{2}}\left(\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} - \begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_{n+j+1}\right)$$

which is the desired formula in (ii). Now, we apply the formula $F_{a+b} = (F_aL_b + F_bL_a)/2$ with $(a, b) = (n, 4m + 2 - j)$ and $(n, j + 1)$, respectively, in order to get

$$\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} = \frac{1}{2}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_nL_{4m+2-j} + \frac{1}{2}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{4m+2-j}L_n. \tag{4.1}$$

$$\begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_{n+j+1} = \frac{1}{2}\begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_nL_{j+1} + \frac{1}{2}\begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_{j+1}L_n. \tag{4.2}$$

Taking $F_{4m+2-j}\left[{4m+2 \atop j}\right]_F = F_{j+1}\left[{4m+2 \atop j+1}\right]_F$, the identity (4.1) becomes

$$\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_{n+4m+2-j} = \frac{1}{2}\begin{bmatrix}4m+2\\ j\end{bmatrix}_F F_nL_{4m+2-j} + \frac{1}{2}\begin{bmatrix}4m+2\\ j+1\end{bmatrix}_F F_{j+1}L_n. \tag{4.3}$$

We substitute (4.2) and (4.3) in (ii) yielding

$$F_n = \sum_{j=0}^{2m} (-1)^{\frac{j(j+1)}{2}} \left(\frac{1}{2} \begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F F_n L_{4m+2-j} - \frac{1}{2} \begin{bmatrix} 4m+2 \\ j+1 \end{bmatrix}_F F_n L_{j+1} \right).$$

Thus,

$$2 = \sum_{j=0}^{2m} (-1)^{\frac{j(j+1)}{2}} \left(\begin{bmatrix} 4m+2 \\ j \end{bmatrix}_F L_{4m+2-j} - \begin{bmatrix} 4m+2 \\ j+1 \end{bmatrix}_F L_{j+1} \right)$$

which completes the proof. □

Now, we denote

$$\sigma(n) = \sum_{j=0}^{4m} (-1)^{\frac{j}{2}(j-1)} \begin{bmatrix} 4m \\ j \end{bmatrix}_F F_{n+4m-j}$$

and the sum of positive summands and the sum of negative summands of $\sigma(n)$, respectively, by

$$\sigma_P(n) = \sum_{\substack{j \in \{0,\dots,4m\} \\ j(j-1) \equiv 0 \pmod 4}} \begin{bmatrix} 4m \\ j \end{bmatrix}_F F_{n+4m-j},$$

$$\sigma_N(n) = \sum_{\substack{j \in \{0,\dots,4m\} \\ j(j-1) \equiv 2 \pmod 4}} \begin{bmatrix} 4m \\ j \end{bmatrix}_F F_{n+4m-j}.$$

Hence,

$$\sigma_P(n) = \sum_{l=0}^{m} \begin{bmatrix} 4m \\ 4l \end{bmatrix}_F F_{n+4m-4l} + \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+1 \end{bmatrix}_F F_{n+4m-(4l+1)}$$

and

$$\sigma_N(n) = \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+2 \end{bmatrix}_F F_{n+4m-(4l+2)} + \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+3 \end{bmatrix}_F F_{n+4m-(4l+3)}.$$

Further we denote

$$\sigma_{P_1}(n) = \sum_{l=0}^{m} \begin{bmatrix} 4m \\ 4l \end{bmatrix}_F F_{n+4m-4l}, \ \sigma_{P_2}(n) = \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+1 \end{bmatrix}_F F_{n+4m-(4l+1)},$$

$$\sigma_{N_1}(n) = \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+2 \end{bmatrix}_F F_{n+4m-(4l+2)}, \ \sigma_{N_2}(n) = \sum_{l=0}^{m-1} \begin{bmatrix} 4m \\ 4l+3 \end{bmatrix}_F F_{n+4m-(4l+3)}.$$

Corollary 1. *Let m, n be any positive integers. Then*

$$\sigma_{P_1}(n) + \sigma_{P_2}(n) - \sigma_{N_1}(n) - \sigma_{N_2}(n) = 0. \tag{4.4}$$

Proof. Identity (4.4) follows from the fact that $\sigma(n) = \sigma_P(n) - \sigma_N(n)$ and identity (1.1). □

Conjecture 1. *Let m, n be any positive integers. Then*

$$\sigma_{P_1}(n)+\sigma_{P_2}(n)=F_{4m+n}\prod_{i=1}^{2m-1}L_{2i}, \tag{4.5}$$

$$\sigma_{P_1}(n)-\sigma_{N_1}(n)=(-1)^m F_{2m+n}L_{2m}\prod_{i=1}^{2m-1}L_i^2,$$

$$\sigma_{P_1}(n)-\sigma_{N_2}(n)=F_n\prod_{i=1}^{2m-1}L_{2i}.$$

Corollary 2. *Let m, n be any positive integers. Then*

$$\sigma_{P_1}(n)=\frac{1}{2}F_{2m+n}L_{2m}\left((-1)^m\prod_{i=1}^{2m-1}L_i^2+\prod_{i=1}^{2m-1}L_{2i}\right),$$

$$\sigma_{P_2}(n)=\frac{1}{2}\left((-1)^{m+1}F_{2m+n}L_{2m}\prod_{i=1}^{2m-1}L_i^2+L_{2m+n}F_{2m}\prod_{i=1}^{2m-1}L_{2i}\right),$$

$$\sigma_{N_1}(n)=\frac{1}{2}F_{2m+n}L_{2m}\left((-1)^{m+1}\prod_{i=1}^{2m-1}L_i^2+\prod_{i=1}^{2m-1}L_{2i}\right),$$

$$\sigma_{N_2}(n)=\frac{1}{2}\left((-1)^{m}F_{2m+n}L_{2m}\prod_{i=1}^{2m-1}L_i^2+L_{2m+n}F_{2m}\prod_{i=1}^{2m-1}L_{2i}\right).$$

Proof. Solving the system of linear equations in (4.4) and (4.5) we clearly obtain assertion. □

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MSC2010: 11B39

Departamento de Matemática, Universidade de Brasília, Brasília, DF, 70910-900, Brazil
E-mail address: diego@mat.unb.br

Department of Mathematics, University of Hradec Králové, Rokitanského 62, Hradec Králové, 500 03, Czech Republic
E-mail address: Pavel.Trojovsky@uhk.cz