LIMITS OF POLYNOMIAL SEQUENCES

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ABSTRACT. Certain sequences of recursively defined polynomials have limiting power series. This fact is proved for a class of second-order recurrences, and the problem for higher order recurrences is stated.

We begin with an example and then generalize. Let $p_0(x) = 1$, $p_1(x) = 1 + x$, and

$$p_n(x) = -xp_{n-1}(x) + (x^2 + 2x)p_{n-2}(x) + x + 1 \tag{1}$$

for $n \geq 2$. Polynomials determined by these conditions are shown here:

$$\begin{aligned}
&p_0(x) = 1\\
&p_1(x) = 1 + x\\
&p_2(x) = 1 + 2x\\
&p_3(x) = 1 + 2x + x^2 + x^3\\
&p_4(x) = 1 + 2x + 3x^2 + x^3 - x^4\\
&p_5(x) = 1 + 2x + 3x^2 + x^3 + 2x^4 + 2x^5\\
&p_6(x) = 1 + 2x + 3x^2 + 5x^3 + 4x^4 - 3x^5 - 3x^6\\
&p_7(x) = 1 + 2x + 3x^2 + 5x^3 + x^5 + 9x^6 + 5x^7\\
&p_8(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + 13x^5 - 3x^6 - 18x^7 - 8x^8.
\end{aligned}$$

The list suggests that the polynomials "approach" a limiting series. The purpose of this note is to examine such limiting behavior.

Throughout, all polynomials have integer coefficients. The expression "$\lim_{n\to\infty} p_n$ exists" is defined from (2) as follows: for every $k \geq 0$, there exists N such that if $n \geq N$, then $p(n+1, k) = p(n, k)$. That is, the coefficient of x^k in $p_n(x)$ eventually becomes constant. Writing that common coefficient as s_k and putting

$$S(x) = s_0 + s_1x + s_2x^2 + \cdots$$

gives

$$\lim_{n\to\infty} p_n = S.$$

For the example above, the limiting coefficients are Fibonacci numbers, and

$$S(x) = \frac{1 + x}{1 - x - x^2}.$$

To generalize, suppose that

$$p_n = p_n(x) = p(n, 0) + p(n, 1)x + \cdots + p(n, n)x^n \tag{2}$$

are polynomials given by $p_0(x) = r$, $p_1(x) = sx + t$, and

$$p_n(x) = (ax + b)p_{n-1}(x) + (cx^2 + dx + e)p_{n-2}(x) + fx + g \tag{3}$$

for $n \geq 2$, where $a \neq 0$. For each $n \geq 0$, we seek recurrence relations for the numerical sequence $p(n, k)$, for $k = 0, 1, 2, \ldots$. These coefficients $p(n, k)$ are related to derivatives of $p_n(x)$ by Cauchy's formula,

$$p(n, k) = p_n^{(k)}(0)/k! \tag{4}$$

First,

$$p_n' = ap_{n-1} + dp_{n-2} + bp_{n-1}' + ep_{n-2}' + f + x(ap_{n-1}' + 2cp_{n-2} + dp_{n-2}') + cx^2p_{n-2}', \tag{5}$$

from which it follows inductively that

$$\begin{aligned} p_n^{(k)} &= k(ap_{n-1}^{(k-1)} + dp_{n-2}^{(k-1)} + (k-1)cp_{n-2}^{(k-2)}) + bp_{n-1}^{(k)} + ep_{n-2}^{(k)} \\ &\quad + x(ap_{n-1}^{(k)} + 2kcp_{n-2}^{(k-1)} + dp_{n-2}^{(k)}) + cx^2p_{n-1}^{(k)} \end{aligned} \tag{6}$$

for $k \geq 2$. Putting $x = 0$ in (6) and applying (4),

$$\begin{aligned} p(n, k) &= ap(n-1, k-1) + dp(n-2, k-1) + cp(n-2, k-2) \\ &\quad + bp(n-1, k) + ep(n-2, k) \end{aligned}$$

for $n \geq 2$ and $k \geq 2$. Initial values are given by

$$\begin{aligned} &p(0, 0) = r,\ p(1, 0) = t,\ p(1, 1) = s, \\ &p(2, 0) = bt + er + g, \\ &p(2, 1) = at + dr + bs + f, \end{aligned}$$

and, for $n \geq 3$,

$$p(n, 1) = ap(n-1, 0) + dp(n-2, 0) + bp(n-1, 1) + ep(n-2, 1) + f. \tag{7}$$

Suppose now that $b = e = 0$ in (3). Then by (7),

$$p(n, 1) = ap(n-1, 0) + dp(n-2, 0) + f. \tag{8}$$

Also, $p(n, 0) = g$ for all $n \geq 2$, by (3), and $p(n, 1) = p_n'(0) = ag + dg + f$ for all $n \geq 4$, by (5). Consequently, by (8),

$$p(n, 2) = (a + d)(ag + dg + f) + cg$$

for all $n \geq 6$. Inductively, therefore, by (8), the coefficient $p(n, k)$ is constant for all $n \geq 2k+2$, for all $k \geq 0$. Accordingly, $\lim_{n\to\infty} p_n$ exists, and substituting $S(x)$ for each of $p_n(x)$, $p_{n-1}(x)$, and $p_{n-2}(x)$ in (3) yields

$$S(x) = \frac{g + fx}{1 - (a + d)x - cx^2}.$$

We close with questions.

(1) Can $\lim_{n\to\infty} p_n$ exist when b and e are not both 0?

(2) Do these results generalize for recurrences of higher order? Specifically, if $m \geq 3$ and polynomials $p_n(x)$ satisfy a recurrence

$$p_n(x) = q_1(x)p_{n-1}(x) + \cdots + q_m(x)p_{n-m}(x) + r_m(x),$$

where $q_i(x)$ is a polynomial of degree i for $1 \leq i \leq m$ and $r_m(x)$ is a polynomial of degree $m-1$, then what conditions on the polynomials $q_i(x)$ ensure that $\lim_{n\to\infty} p_n$ exists?

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