AN ANALOGUE OF THE DUCCI SEQUENCES OVER FUNCTION FIELDS

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Abstract. A classical Ducci sequence of integers is a sequence of n-tuples of integers obtained by iterating the map $(a_1, \ldots, a_n) \mapsto (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_n - a_1|)$. In this paper, we study a natural analogue of the Ducci sequences defined over function fields that are motivated by the number field-function field analogy. Results that are analogous to the classical case have been found, and differences between the two cases are also explored.

1. Introduction

Let $d \geq 3$ be an integer. A classical Ducci sequence of integers is a sequence of d-tuples $\mathbf{u}, \tilde{D}(\mathbf{u}), \tilde{D}^2(\mathbf{u}) = \tilde{D}(\tilde{D}(\mathbf{u})), \ldots$, obtained by iterating the map $\tilde{D} : \mathbb{Z}^d \to \mathbb{Z}^d$, where

$$\tilde{D}(u_0, u_1, \ldots, u_{d-1}) = (|u_0 - u_1|, |u_1 - u_2|, \ldots, |u_{d-1} - u_0|). \tag{1.1}$$

The origin of this sequence dates back to E. Ducci, who is credited in [12] for discovering the fact that every Ducci sequence will eventually stabilize at the zero vector when $d = 4$. In fact, the same property holds if and only if d is a power of 2. For any positive integer d, the dynamic system induced by D always forms a cycle. Having relations to the cyclotomic polynomials, the Wieferich primes [4] and also the well-known conjecture of Gilbreath [15, 16, 18], these cycles have many interesting properties and are a popular object to study [4, 5, 8, 9, 13, 14, 17]. It turns out that any d-tuples in a cycle are constant multiples of binary tuples, i.e. tuples with entries in $\{0, 1\}$. In this case, the Ducci map is essentially the following linear transformation in the vector space $\mathbb{F}_2^d$:

$$\tilde{D}(u_0, \ldots, u_{d-1}) = (u_0 + u_1, \ldots, u_{d-1} + u_0). \tag{1.2}$$

In the realm of number fields, the study of Ducci sequences have been generalized to the algebraic numbers [10] and the real numbers [6, 7]. For other generalizations, see for example [1, 2, 3, 11, 21, 22].

There is a well-known analogy between number fields and function fields of one variable over finite fields. Motivated by this analogy, in this paper we initiate a study of a natural analogue of the Ducci sequences over function fields based on equation (1.1) (see Definition 3.1 in Section 3).

As in the number field case, the dynamics induced in our case will always form cycles. However, we will see that those cycles need not be constant multiples of binary tuples. Therefore the standard strategy of studying Ducci sequences using cyclotomic polynomials is not applicable. Nevertheless, we are able to show that our Ducci sequences exhibit similar behavior as their counterpart in the number field case. In particular, the famous property that a Ducci sequence of d-tuples always ends in the zero cycle if and only if d is a power of 2 is also true in our case. On the other hand, when we extend our consideration to Ducci sequences over the power series ring, they behave quite differently from the classical Ducci sequences over $\mathbb{R}$. We will also consider periods of the cycles. One striking difference from the number field case is that the characteristic of the constant field plays an important role on the length of the cycles, which makes such cycles more mysterious than the ones over number fields.

It is hardly surprising that Ducci sequences over function fields are intimately related to the one over number fields. In fact, one of our main ideas is to transfer the questions in the function field case to the corresponding questions in the number field case, which are studied more extensively. We remark that there are other generalizations that work in the case of polynomial rings (see for example [1, 2]), but their generalizations are based on the homomorphism (1.2) and are therefore different from our generalization.

The paper is organized as follows. In Section 2 we recall some basics of the theory of function fields that are needed to define our Ducci sequences. In Section 3 we define our Ducci sequences and show that they form cycles just like the usual Ducci sequence over $\mathbb{Z}$. In Section 4 and Section 5, function field analogues of the results in [7, 12] are obtained. In particular, we prove that all d-tuples are vanishing if and only if d is a power of 2. In Section 4, we also study other vanishing sequences and obtain a bound on the vanishing time. We end with a discussion on the cycles generated by the Ducci sequences in Section 6. Some possible directions for further research are also discussed throughout the paper.

2. Function Fields

Let q be a power of a prime p. By a *function field* we mean a function field of one variable over a finite field, or equivalently a finite extension of some $\mathbb{F}_q(x)$. In this section we collect some facts about function fields which we will use later, with the number field-function field analogy in mind. The main references are the books [19] and [20].

Let K be a function field with field of constants $\mathbb{F}_q$. Fix an arbitrary degree one prime P_∞ of K (which serves the purpose of an "infinity prime"), and let v_∞ be its corresponding (normalized) valuation. Suppose S is a finite set of primes in K containing P_∞, the ring of S-integers is defined by

$$\mathcal{O}_S = \{x \in K : v_P(x) \geq 0 \text{ for all } P \notin S\}.$$

This is the ring analogous to the ring of S-integers in a number field. To simplify notations, when $S = \{P_\infty\}$, we will write

$$A_K := \{x \in K : v_P(x) \geq 0 \text{ for all } P \neq P_\infty\}.$$

Next, we need an analogue of $\mathbb{R}$ in the function field case. This can be done by completion. Let K_∞ be the completion of K at P_∞ (i.e. with respect to v_∞). If π is any uniformizer at P_∞ (i.e. $v_\infty(\pi) = 1$), since P_∞ has degree one, we have that K_∞ is isomorphic to $\mathbb{F}_q((\pi))$, the field of formal Laurent series in T. The following example is one which the readers should bear in mind as we will frequently return to this case for illustrating our results concretely.

Example 2.1. *The simplest and most concrete example of a function field is the field $K = \mathbb{F}_q(x)$. Let P_∞ be the prime corresponding to the pole of x, and $S = \{P_\infty\}$. The valuation v_∞ corresponding to P_∞ can be characterized as follows: if f and g are polynomials and $f/g \in K$, then $v_\infty(f/g) = \deg g - \deg f$. The ring of integers A_K is the polynomial ring $\mathbb{F}_q[x]$, which is an analogue of $\mathbb{Z}$ in the number field case. Choose the uniformizer $\pi = 1/x$ at P_∞, then the field $K_\infty = \mathbb{F}_q((\pi))$ is an analogue of $\mathbb{R}$.*

Remark 2.2. *Unlike the number field case for which we have to isolate the archimedean primes, in the function field case we are free to choose any degree one prime P_∞ to serve as the "infinity prime". As far as the chosen prime is of degree one, different choices of P_∞ will give us isomorphic rings, and therefore the same theory.*

By a small modification, it is also possible to extend our consideration to allow the case when P_∞ can be of arbitrary degree, but we will only consider the case of $\deg P_\infty = 1$ *here for the sake of simplicity.*

Now we have the rings that we intend to work on. To define the analogue of the Ducci sequence using (1.1), we will also need an analogue of the (archimedean) absolute value. Before we can define such an analogue, the first thing we need is the notion of positivity in function fields.

Fix a uniformizer π at the prime P_∞, then K_∞^* has the decomposition

$$K_\infty^* \cong \mathbb{F}_q \times \pi^{\mathbb{Z}} \times U_{1,\pi},$$

where $U_{1,\pi}$ is the group of 1-units modulo π:

$$U_{1,\pi} = \{x \in K_\infty : x \equiv 1 \pmod{\pi}\}.$$

Thus, every element $x \in K_\infty^*$ has a unique decomposition

$$x = \operatorname{sgn}(x)\pi^{v_\infty(x)}u(x) \tag{2.1}$$

for some $\operatorname{sgn}(x) \in \mathbb{F}_q$ and $u(x) \in U_{1,\pi}$. Note that the decomposition depends on the choice of uniformizer π, but it is good enough for our purpose. For any $x \in K_\infty^*$, we define the *sign* of x to be $\operatorname{sgn}(x)$, and say that x is *positive* if $\operatorname{sgn}(x) = 1$. In particular, we get a notion of positivity in K^* by restriction as $K \subseteq K_\infty$. It is worth noting that there are two signs in the number field case, namely $+1$ and -1. They are also the only units in $\mathbb{Z}^*$. In the function field case, the units in $\mathbb{F}_q[x]^*$ are elements of $\mathbb{F}_q^*$, which are exactly the $q-1$ possible signs in this case. We are now ready to define our absolute value.

Definition 2.3. *Let K be a function field and K_∞ be as above. For any $x \in K_\infty^*$, write $x = \operatorname{sgn}(x)\pi^{v_\infty(x)}u(x)$ as in* (2.1)*. Then the absolute value of x is defined by*

$$|x| = \pi^{v_\infty(x)}u(x).$$

We extend our definition of the absolute value to the whole K_∞ by setting $|0| = 0$.

This is to say that we multiply x with a suitable sign element to make it sign 1, which is exactly what the absolute value in the number field case does. Note that this definition actually depends on the choice of π, but different choices of π will give isomorphic theories.

Example 2.4. *We return to the case of $K = \mathbb{F}_q(x)$, P_∞ is the pole of x, $S = \{P_\infty\}$ and $\pi = 1/x$. The field K_∞ is isomorphic to the field of Laurent series $\mathbb{F}_q((\pi))$. For an element $f = \sum_{i=-m}^{\infty} a_i\pi^i \in K_\infty^*$ with $a_{-m} \neq 0$, it is not difficult to see that f is positive if and only if $a_{-m} = 1$. In particular, a polynomial $f(x) \in K$ is positive if and only if it is monic.*

The absolute value in K is to multiply a suitable element in $\mathbb{F}_q^*$ to make the polynomial monic. For example, if q is odd, then

$$\left|2x^2 + x + 3\right| = x^2 + \frac{q+1}{2}x + \frac{3(q+1)}{2}.$$

The last thing we need is some preliminaries on divisors. The group of divisors of K, denoted by $\mathcal{D}_K$, is the free abelian group generated by the primes. Thus a divisor of K can be written as a finite formal sum $D = \sum_P a_P P$. Here the sum runs through all the primes P in K, but only finitely many a_P can be nonzero. The *degree* of D is $\deg D = \sum_P a_P \deg P$. The divisor D is *effective* if all $a_P \geq 0$, and we write $D \geq 0$ in this case. For a function $f \in K^*$, we can associate to it the divisor $(f) = \sum_P v_P(f)P$. Thus the divisor (f) keeps track of the orders of zeros and poles of f. One can show that $\deg((f)) = 0$ for any $f \in K^*$. Likewise, the *zero divisor* and *pole divisor* of f is defined by

$$(f)_0 = \sum_{P, v_P(f)>0} v_P(f)P, \qquad (f)_\infty = -\sum_{P, v_P(f)<0} v_P(f)P.$$

Given two divisors $D = \sum_P a_P P$, $E = \sum_P b_P P$, we define the GCD and LCM of them to be

$$GCD(D,E) = \sum_P \min\{a_P, b_P\}P, \qquad LCM(D,E) = \sum_P \max\{a_P, b_P\}P.$$

Let D be a divisor. Define the set

$$L(D) = \{x \in K^* : (x) + D \geq 0.\}. \tag{2.2}$$

Thus, $L(D)$ is the set containing all functions in K such that the pole order at each prime does not exceed (or zero order is not less than) a certain number prescribed in the coefficients of D. The fact about $L(D)$ that we need in this paper is the following.

Lemma 2.5. *The set $L(D)$ is a finite dimensional vector space over $\mathbb{F}_q$. In particular, it is a finite set. In addition, the set $L(D)$ is nonempty if $\deg D$ is sufficiently large.*

Remark 2.6. *It is possible to calculate the exact dimension $\ell(D)$ of the space $L(D)$ over $\mathbb{F}_q$ using the famous Riemann-Roch Theorem (see for example* [19, Theorem 5.4]*) and a little bit more algebraic geometry, but we will not need it in our paper.*

3. Ducci Sequences in Function Fields

We now define our analogue of the Ducci sequences over function fields.

Definition 3.1. *Let K be a function field and P_∞ be a prime of degree one. Let K_∞ be the completion of K at P_∞, and fix a uniformizer π at P_∞. A Ducci sequence over K_∞ is a sequence of d-tuples $u, D(u), D^2(u) = D(D(u)), \ldots$ obtained by iterating the map $D : K_\infty^d \to K_\infty^d$, where*

$$D(u_0, u_1, \ldots, u_{d-1}) = (|u_0 - u_1|, |u_1 - u_2|, \ldots, |u_{d-1} - u_0|). \tag{3.1}$$

Here the absolute value is as in Definition 2.3.

Remark 3.2. *It is easy to see that every Ducci sequence over K can be regarded as a Ducci sequence over $\mathcal{O}_S$ for some suitable S since there are only finitely many coordinates. In fact, we can do better. Suppose we start with the vector $\mathbf{u} = (u_0, u_1, \ldots, u_{d-1})$. By Lemma 2.5, we can find a function f that has only a pole (of some very high order) at P_∞ and prescribed zeros so that fu_i has no poles except at P_∞. We can also make f positive by multiplying it with an appropriate scalar. It is clear that the properties of $\mathbf{u}$ can be deduced from that of $f\mathbf{u}$, and so we reduce to considering the case of Ducci sequences over A_K. This is analogous to the fact that we can reduce Ducci sequences over $\mathbb{Q}$ to those over the integers by clearing denominators.*

Example 3.3. *Again we return to the case of $K = \mathbb{F}_q(x)$, P_∞ is the pole of x and $\pi = 1/x$. Suppose we start with the vector $\mathbf{u} = (0, 1, x, x^2)$. If q is not of characteristic 2, the Ducci sequence generated by $\mathbf{u}$ is*

$$\begin{aligned}&(0,1,x,x^2), (1, x-1, x^2-x, x^2), (x-2, x^2-2x+1, x, x^2-1),\\&(x^2-3x+3, x^2-3x+1, x^2-x-1, x^2-x+1), (1, x-1, 1, x-1),\\&(x-2, x-2, x-2, x-2), (0,0,0,0), (0,0,0,0), \ldots.\end{aligned}$$

If K is of characteristic 2, the Ducci sequence generated by $\mathbf{u}$ is slightly different:

$$(0,1,x,x^2),(1,x-1,x^2-x,x^2),(x,x^2+1,x,x^2-1),$$
$$(x^2-x+1,x^2-x+1,x^2-x-1,x^2-x+1),(1,0,1,0),$$
$$(1,1,1,1),(0,0,0,0),(0,0,0,0),\dots.$$

Note that the sequence vanishes (i.e. ends in the zero cycle). This is true in general when the number of coordinates is a power of 2 (see Theorem 4.1 in the next section).

Example 3.4. *The above example is a vanishing one. Now we look at a non-vanishing example. Let $K=\mathbb{F}_q(x)$, P_∞ be the pole of x and $\mathbf{u}=(0,1,x)$. The Ducci sequence generated by $\mathbf{u}$ reads*

$$(0,1,x),(1,x-1,x),(x-2,1,x-1),(x-3,x-2,1),(1,x-4,x-3),\dots.$$

Thus, $D^4(\mathbf{u})=D(\mathbf{u})-(0,3,3)$ and more generally $D^{3n+1}(\mathbf{u})=D(\mathbf{u})-(0,3n,3n)$. Therefore the sequence eventually forms a cycle. However, unlike the number field case, the cycle is not binary.

Another important observation is that the period of this cycle depends on the characteristic: in characteristic 3 this sequence has period 3, but in any other characteristic $p\neq 3$, the sequence has period $3p$. For example, the period of $\mathbf{u}$ is $LCM(3,p)$. We will have a more detailed discussion on the cycles and their periods in Section 6.

From now on, by saying "Ducci sequences" without a specific domain we mean the one in the function field case. The first thing we will prove about the Ducci sequences is that over K they always form cycles. This is analogous to the fact that Ducci sequences over $\mathbb{Q}$ always form cycles.

Proposition 3.5. *Every Ducci sequence over K eventually forms a cycle. For example, for any $\mathbf{u}\in K^d$, there exists positive integers n_0, k such that $D^n(\mathbf{u})=D^{n+k}(\mathbf{u})$ for all $n\geq n_0$.*

Proof. Suppose we start with the vector $\mathbf{u}=(u_0,\dots,u_{d-1})$. Since $v_P(f)=v_P(|f|)$ and $v_P(f-g)\geq\min\{v_P(f),v_P(g)\}$, the valuations of the entries appearing in the Ducci sequence generated by $\mathbf{u}$ are bounded below by that of $\mathbf{u}$. In particular, if we set $D=GCD((u_0)_\infty,(u_1)_\infty,\dots,(u_{d-1})_\infty)$, then every entry appearing in the Ducci sequence is in $L(D)$. By Lemma 2.5, $L(D)$ is a finite set. Therefore there are only finitely many possible vectors in the Ducci sequence, which means that it must eventually form a cycle. □

4. Vanishing Ducci Sequences

In this section we investigate vanishing tuples, i.e. tuples for which their generated Ducci sequences stabilize at the zero vector. Our first result is the function field analogue of the celebrated theorem that every Ducci sequence of d-tuples vanishes if and only if d is a power of 2.

Theorem 4.1. *Let D be the Ducci map as in Definition 3.1. The following are equivalent:*

(1) *d is a power of 2,*
(2) *For all $x\in K^d$, $D^n(x)=(0,0,\dots,0)$ for all sufficiently large n.*

Proof. By Remark 3.2, we may assume the Ducci sequence is over A_K. Suppose we start with the vector $\mathbf{u}=(u_0,u_1,\dots,u_{d-1})$, where by our assumption, all u_i's only have poles at P_∞. We also assume that all u_i's are positive (if not, then apply D once and all entries will be positive).

Let $m_i \geq 0$ be such that $v_\infty(u_i) = -m_i$ (and $m_i = -\infty$ if $u_i = 0$). Let $m = \max_{0\leq i\leq d-1}\{m_i\}$. We proceed by induction on m. The case $m = -\infty$ is trivial. If $m = 0$, then all entries of $\mathbf{u}$ are constants, so after applying D once we end up with zeros and ones. The theory of Ducci sequences in the classical case then gives the desired result.

Suppose now $m > 0$. Define $\phi : K \to \mathbb{Z}$ to be the following map:

$$\phi(u_i) = \begin{cases} 1, & \text{if } m_i = m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

We extend ϕ to $\phi : K^d \to \mathbb{Z}^d$ in the natural way.

Since all u_i are positive, if u_i and u_j have the same valuation at P_∞, say they have the decomposition (see (2.1))

$$u_i = \pi^{-m_i} x_i \quad \text{and} \quad u_j = \pi^{-m_i} x_j.$$

Then $|u_i - u_j| = \pi^{-m_i}(x_i - x_j)$ has a less negative valuation since both x_i and x_j are 1-units, meaning that π divides their difference. Thus we have

$$\mathbf{u}' = D(\mathbf{u}) \Rightarrow \phi(\mathbf{u}') = \tilde{D}(\phi(\mathbf{u})). \tag{4.2}$$

Here D is the Ducci map over K, and $\tilde{D}$ is the one over the integers. By the theory in the classical case, if d is a power of 2, the Ducci sequence generated by $\phi(\mathbf{u})$ will vanish. This means at that point the corresponding Ducci sequence over K has entries with valuation strictly larger than $-m$. This completes the induction when d is a power of 2.

If d is not a power of 2, there are non-vanishing Ducci sequences over $\mathbb{F}_2$. Those sequences can be regarded as non-vanishing sequences in K with valuation zero. Non-vanishing sequences with higher valuation can be easily constructed via ϕ. This completes the proof of our theorem. □

We continue our investigation on vanishing d-tuples. Now we look at the case when d is not a power of 2. First we need some preliminaries on cyclotomic polynomials, which is an important tool for studying the classical Ducci sequences. We will confine ourselves to the case of Ducci sequences over $\mathbb{F}_2$ since this is what we need.

We identify the d-tuple $\mathbf{u} = (u_0, \dots, u_{d-1})$ with the polynomial $f_{\mathbf{u}}(x) = u_0 x^{d-1} + u_1 x^{d-2} + \dots + u_{d-1}$ in the ring $R_d = \mathbb{F}_2[x]/(x^d - 1)$. The Ducci map (1.2) can then be identified with the multiplication by $1 + x$. Write $d = 2^s t$ for t odd, over $\mathbb{F}_2$ we have the factorization

$$x^d - 1 = (x^{2^s})^t - 1 = \prod_{l|t} \Phi_l(x^{2^s}) = \prod_{l|t} \Phi_l(x^{2^s}).$$

Here $\Phi_l(x)$ is the lth cyclotomic polynomial. Factorize $\Phi_l(x) = \prod_{i=1}^{r_l} \Phi_{l,i}(x)$ into a product of irreducible polynomials. By the Chinese Remainder Theorem, we have

$$R_d = \frac{\mathbb{F}_2[x]}{\left(\prod_{l|t} \Phi_l(x^{2^s})\right)} = \prod_{l|t}\prod_{i=1}^{r_l} \frac{\mathbb{F}_2[x]}{(\Phi_{l,i}(x))^{2^s}} =: \prod_{l|t}\prod_{i=1}^{r_l} R_{l,i}. \tag{4.3}$$

Each $R_{l,i}$ is a local ring with maximal ideal generated by $\Phi_{l,i}(x)$. We also have $r_1 = 1$, $R_{1,1} = \mathbb{F}_2[x]/(x+1)^{2^s}$ and $(x + 1)$ is not a factor of $\Phi_{l,i}$ for any $l \neq 1$.

The following lemma is a slight generalization of [4, Theorem 3.2(1)] in the case of $\mathbb{F}_2$.

Lemma 4.2. *Let $d = 2^s t$ with t odd. Let $\mathbf{u} = (u_0, \dots, u_{d-1}) \in \mathbb{F}_2^d$, then $\mathbf{u}$ vanishes if and only if $u_i = u_{i+2^s}$ for all i (here the index is taken modulo d).*

Proof. We will follow the idea of the proof in [4, Theorem 3.2]. Since all $R_{l,i}$ are local, every element in $R_{l,i}$ is either invertible or nilpotent. The tuple $\mathbf{u}$ vanishes if and only if $f_{\mathbf{u}}(x) = 0$ in those $R_{l,i}$ in which $x+1$ is invertible. In our case, $x+1$ is nilpotent in $R_{1,1} = \mathbb{F}_2[x]/(x+1)^{2^s}$ and is invertible in all other $R_{l,i}$ (since $x+1$ does not divide other $\Phi_{l,i}(x)$). So $\mathbf{u}$ vanishes if and only if $f_{\mathbf{u}}(x)$ is a multiple of $(x^d-1)/(x+1)^{2^s} = 1 + x^{2^s} + \ldots + x^{2^s(t-1)}$. The result follows. □

Now we are able to give some criterion for a d-tuple to vanish (or non-vanish).

Proposition 4.3. *Let $d = 2^s t$ with t odd, and $\mathbf{u} = (u_0, \ldots, u_{d-1}) \in K^d$.*

1. *Let ϕ be as in (4.1). If $\phi(\mathbf{u})$ is non-vanishing (in $\mathbb{F}_2$), then $\mathbf{u}$ is non-vanishing.*
2. *If $u_i = u_{i+2^s}$ for all i, then it is vanishing.*

Proof. Using Remark 3.2 we can reduce to the case of Ducci sequences over A_K. Part (1) is then trivial in view of the definition of ϕ and the equation (4.2). For part (2), let $D^n(\mathbf{u}) = (u_{n,0}, \ldots, u_{n,d-1})$. Note that $u_i = u_{i+2^s}$ implies that $u_{n,i} = u_{n,i+2^s}$ for all i, n. The result then follows from (4.2) and Lemma 4.2, using an induction argument similar to the proof of Theorem 4.1. □

Remark 4.4. *In general, part (2) of the above proposition does not exhaust all vanishing tuples. For example, if $K = \mathbb{F}_q(x)$ with $q \geq 3$. Suppose $0, 1, \alpha$ are distinct elements in $\mathbb{F}_q$, then the tuple $(x, x-1, x-\alpha)$ is vanishing but is not of the above type. However, the tuple $(x, x-1, x-1)$, which is very similar to the previous one, is non-vanishing. It will be interesting to find all vanishing tuples for a general d, which we propose as a challenging problem.*

Our final result in this section is a bound on the length for the vanishing tuples. Before we state our bound, we make precise the meaning of the length.

Definition 4.5. *Let $\mathbf{u} \in K^d$ be a vanishing tuple. The length of $\mathbf{u}$ is the positive integer n so that $D^n(\mathbf{u}) = 0$ but $D^{n-1}(\mathbf{u}) \neq 0$.*

By Remark 3.2 it suffices to consider the case over A_K, and we can assume $\mathbf{u}$ to be positive (if $\mathbf{u}$ is not positive the length increases by at most one).

Proposition 4.6. *Let $d = 2^s t$ and $\mathbf{u} = (u_0, \ldots, u_{d-1}) \in K^d$ be a positive, vanishing d-tuple such that all u_i only have poles at P_∞. Let $m = \max_i(-v_\infty(u_i))$, then the length of d is at most $2^s(m+1)$.*

Proof. If $\mathbf{u}$ is a constant vector, by (4.3) and the idea of proof in Lemma 4.2, it follows that the length is at most 2^s. Now the bound follows by an induction argument using the map ϕ as in the proof of Theorem 4.1. □

Remark 4.7. *The upper bound in the above proposition can be achieved. For example when $\mathbf{u} = (1, 0, \ldots, 0)$ is a constant vector with 2^s coordinates, then its length is 2^s. For a more non-trivial example, consider $K = \mathbb{F}_q(x)$ with $q \geq 3$, P_∞ the pole of x. Let $0, 1, \alpha$ be three distinct elements in $\mathbb{F}_q$. Then the tuple $\mathbf{u} = (x, x^2+\alpha, x+1, x^2+x, x+\alpha, x^2+1)$ has length* 6.

However, experimental data suggests that the above upper bound is far from optimal when m is large. In fact, if the genus of K is nonzero, there is no function $f \in K$ with only a simple pole at P_∞, and the upper bound (for non-constant $\mathbf{u}$) can be improved to $2^s m$. Using Riemann-Roch, the upper bound can be further improved to $2^s \ell(mP_\infty)$, where $\ell(D)$ is the dimension of the vector space $L(D)$, defined by (2.2).

5. Ducci Sequences Over K_∞

In this section we investigate the behavior of Ducci sequences over K_∞, which serves as the function field analogue of $\mathbb{R}$. Since $K_\infty \cong \mathbb{F}_q((\pi))$, by multiplying a suitable power of π, we reduce ourselves to considering the behavior of Ducci sequences over the power series ring $R := \mathbb{F}_q[[\pi]]$.

Example 5.1. *Let us look at an example of a Ducci sequence over R. Let*

$$\mathbf{u} = \left(\sum_{i=0}^{\infty} \pi^i, \sum_{i=1}^{\infty} \pi^{2i-1}, \sum_{i=1}^{\infty} \pi^{2i} \right).$$

Then

$$\begin{aligned} D(\mathbf{u}) &= \left(\sum_{i=0}^{\infty} \pi^{2i}, \sum_{i=1}^{\infty} (-1)^{i+1} \pi^i, 1 + \sum_{i=1}^{\infty} \pi^{2i-1} \right), \\ D^2(\mathbf{u}) &= \left(1 + \sum_{i=1}^{\infty} \pi^{2i-1}, \sum_{i=0}^{\infty} \pi^{2i}, \sum_{i=1}^{\infty} (-1)^{i+1} \pi^i \right), \\ D^3(\mathbf{u}) &= \left(\sum_{i=1}^{\infty} (-1)^{i+1} \pi^i, 1 + \sum_{i=1}^{\infty} \pi^{2i-1}, \sum_{i=0}^{\infty} \pi^{2i} \right), \\ D^4(\mathbf{u}) &= D(\mathbf{u}), \end{aligned}$$

and we obtain a cycle of length 3.

Let v_∞ be the normalized valuation at P_∞ as usual, then the valuation induces a norm $\|\cdot\|$ on K_∞ by $\|f\| = q^{-v_\infty(f)}$, where q is the number of elements in the constant field of K. It is easy to see that $\|\cdot\|$ is a (non-archimedean) norm. Having the norm we are able to talk about limits and convergence. In particular, a sequence $\{a_n\}$ tends to zero if and only if its valuation tends to (positive) infinity.

The following is an analogue of the main theorem in [7] in our case. Note that our result is actually quite different from the one in [7] since our norm is non-archimedean.

Theorem 5.2. *Let $\mathbf{u} \in K_\infty^d$. Then exactly one of the following happens:*

1. *The sequence $D^n(\mathbf{u})$ tends to zero as $n \to \infty$,*
2. *The sequence $D^n(\mathbf{u})$ is eventually periodic.*

Proof. Without loss of generality, we will work over the power series ring R. Let $\mathbf{u} = (u_0, \ldots, u_{d-1})$ with the u_i being positive. Let $m_i = v_\infty(u_i)$, and $m = \min_i\{m_i\}$. Similar to (4.1), define $\psi : K_\infty \to \mathbb{Z}$ to be the following map:

$$\psi(u_i) = \begin{cases} 1, & \text{if } m_i = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{u}' = D(\mathbf{u}) \Rightarrow \psi(\mathbf{u}') = \tilde{D}(\psi(\mathbf{u})).$$

The $\psi(\mathbf{u})$ either vanishes or forms a cycle after a repeat application of $\tilde{D}$. If it vanishes then m increases, and we repeat the process for the new m. There are two possibilities. Either $\psi(\mathbf{u})$ vanishes at all valuation, so that the sequence $D^n(\mathbf{u})$ tends to zero and we are in case (1); or it eventually forms a cycle at some valuation $\tilde{m}$.

In the second case we may without loss of generality assume that $\psi(\mathbf{u})$ is already in a cycle. Thus, $m = \tilde{m}$, and $\mathbf{u} = (u_0, \dots, u_{d-1})$ with $u_i = c_i\pi^m + w_i$, where $c_i = 0$ or 1 and $w_i \in R$ with $v_\infty(w_i) > m$ for all i. As $v_\infty(w_i) > m$ and the $\psi(\mathbf{u})\pi^m$ forms a cycle with valuation m, we have

$$D^n(\mathbf{u}) = \tilde{D}^n(\psi(\mathbf{u}))\pi^m + (c_0w_0 + \ldots + c_{d-1}w_{d-1})$$

for some $c_i \in \mathbb{F}_q$. Since there are only finitely many possibilities for the c_i, we see that $D^n(\mathbf{u})$ must form a cycle. This is case (2) and the proof is complete. □

Combining the idea of this proof with Theorem 4.1, we immediately get the following corollary.

Corollary 5.3. *If d is a power of 2, then the sequence $D^n(\mathbf{u})$ either tends to zero as $n \to \infty$, or is zero for all sufficiently large n.*

Several remarks are in order.

Remark 5.4. *Observe that the second part of the proof in the above theorem also works (after a slight modification) in Proposition 3.5, but using the Riemann-Roch type argument in that proposition gives a more simple proof. This observation will be used in the proof of Proposition 6.2.*

Remark 5.5. *In fact, we did not find any example in case* (1) *of the above theorem. We are thus led to the following conjecture that every Ducci sequence over K_∞ actually forms a cycle and case* (1) *does not occur.*

Conjecture. *Every Ducci sequence over K_∞ is eventually periodic.*

6. Cycles of Ducci Sequences over K

In this section we consider cycles of Ducci sequences over a function field K. By the *period* of a tuple $\mathbf{u}$ we mean the period of the cycle generated by the Ducci sequence starting with $\mathbf{u}$. The first observation is that the periods of these cycles depend on the characteristic of the field K. We have already seen an example of this phenomenon in Example 3.4, but sometimes the dependence on the characteristic is not so simple. Consider the following example.

Example 6.1. *Let $K = \mathbb{F}_q(x)$, P_∞ be the pole of x and let α be a nonzero element in $\mathbb{F}_q$. Consider $\mathbf{u} = (x^2+1, x^2+\alpha x, x)$. The Ducci sequence generated by $\mathbf{u}$ is*

$$\begin{aligned}
D(\mathbf{u}) &= (x-\alpha^{-1}, x^2+(\alpha-1)x, x^2-x+1),\\
D^2(\mathbf{u}) &= (x^2+(\alpha-2)x+\alpha^{-1}, x-\alpha^{-1}, x^2-2x+(1+\alpha^{-1})),\\
D^3(\mathbf{u}) &= (x^2+(\alpha-3)x+2\alpha^{-1}, x^2-3x+(1+2\alpha^{-1}), x-\alpha^{-1}),\\
D^4(\mathbf{u}) &= (x-\alpha^{-1}, x^2-4x+(1+3\alpha^{-1}), x^2+(\alpha-4)x+3\alpha^{-1}),\\
D^5(\mathbf{u}) &= (x^2-5x+(1+4\alpha^{-1}), x-\alpha^{-1}, x^2+(\alpha-5)x+4\alpha^{-1}),\\
D^6(\mathbf{u}) &= (x^2-6x+(1+5\alpha^{-1}), x^2+(\alpha-6)x+5\alpha^{-1}, x-\alpha^{-1}),\\
D^7(\mathbf{u}) &= (x-\alpha^{-1}, x^2+(\alpha-7)x+6\alpha^{-1}, x^2-7x+(1+6\alpha^{-1})),
\end{aligned}$$

and so on. In particular,

$$\begin{aligned}
D^4(\mathbf{u}) &= D(\mathbf{u}) + (0, -(\alpha+3)x+(3\alpha^{-1}+1), (\alpha-3)x+(3\alpha^{-1}-1)),\\
D^7(\mathbf{u}) &= D(\mathbf{u}) + (0, -6(x-\alpha^{-1}), -6(x-\alpha^{-1})).
\end{aligned}$$

By induction, we get

$$D^{6n+1}(\mathbf{u}) = D(\mathbf{u}) + (0, -6n(x-\alpha^{-1}), -6n(x-\alpha^{-1})).$$

Hence, if p denotes the characteristic of K, then the period of $\mathbf{u}$ is 3 if $\alpha = 1$ and $p = 2$, and is $LCM(6,p)$ otherwise.

Like the number field case, if d is a power of 2, then from Theorem 4.1 we know that the only possible cycle is the zero cycle, and hence the only possible period is 1. However, the periods seem to be much more mysterious than its number field counterpart in general. Let ϕ be as in (4.1). It is not difficult to see that if $\phi(\mathbf{u})$ has period a, then the period of $\mathbf{u}$ is a multiple of a. On the other hand, we have the following upper bound on the period. We remark that this upper bound is very weak and there should be plenty of room for improvement.

Proposition 6.2. *Let K be a function field whose constant field has q elements. If $\mathbf{u} \in K^d$ is such that $\phi(\mathbf{u})$ has period a, then $\mathbf{u}$ has period at most $LCM(a, q^d)$.*

Proof. Without loss of generality we assume $\mathbf{u} = (u_0, \ldots, u_{d-1}) \in A_K^d$. We will use the observation in Remark 5.4. In particular, let $m = \max_i\{-v_\infty(u_i)\}$ and ϕ be as in (4.1). Write $u_i = \pi^{-m} + w_i$ with $0 \geq v_\infty(w_i) > -m$, we have

$$D^n(\mathbf{u}) = \tilde{D}^n(\phi(\mathbf{u}))\pi^{-m} + (c_0w_0 + \cdots + c_{d-1}w_{d-1}).$$

As $\phi(\mathbf{u})$ has period a and $c_0w_0 + \cdots + c_{d-1}w_{d-1}$ has at most q^d distinct values, the result follows. ☐

Many properties about these cycles remain mysterious. For example we did not touch the interesting question of characterizing the cycles: which tuples can appear in a cycle? What are the possible periods if we start with some vector $\mathbf{u}$? How does the characteristic of K affect the periods in general? Since these cycles have more structures than those over number fields, a more detailed study of such cycles would be an interesting further research topic.

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