DEGREE n RELATIVES OF THE GOLDEN RATIO AND RESULTANTS OF THE CORRESPONDING POLYNOMIALS

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Abstract. Computations are given of the resultants $\mathrm{Res}(s_m, s_n)$ of pairs of Selmer polynomials $s_n = s_n(X) = X^n - X - 1$. It is shown that for each fixed $m \in \mathbb{N}$ the sequence of integers $\mathrm{Res}(s_m, s_{m+k}) = \mathrm{Res}(s_m(X), X^k - 1)$ satisfies a simple linear recursion which can be described in terms of higher Lucas sequences.

1. Introduction

The object of this article is to give some computations of the resultants $\mathrm{Res}(s_m, s_n)$ of pairs of Selmer polynomials $s_n = s_n(X) = X^n - X - 1$. In Section 2 the definition and properties of resultants are reviewed and it is shown that $\mathrm{Res}(s_m, s_{m+k}) = \mathrm{Res}(s_m(X), X^k - 1)$. In Section 3, the Spectral Mapping Theorem is used to give formulas for $\mathrm{Res}(s_2, s_{2+k})$ in terms of the Fibonacci sequence and the Lucas sequence. This shows in particular that $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2(X), X^k - 1)$, is a linear recursion sequence of degree 4 whose values are easily described in terms of the Lucas sequence. In Section 4, we give formulas for the values of the simple linear recurrence sequence $\mathrm{Res}(s_3, s_{3+k}) = \mathrm{Res}(s_3(X), X^k - 1)$, $k \geq 1$, which has order 6, in terms of the order 3 analogue of the Lucas sequence (namely the Perrin sequence). In Section 5, it is shown that if D is an integral domain and $f \in D[X]$ is monic and separable of degree n, then there is an associated Lucas sequence L_f having f as its characteristic polynomial, and that $\mathrm{Res}(f(X), X^k - 1)$, $k \geq 1$, is a simple linear recurrence sequence, whose order can be 2^n, but its values are given in terms of the Lucas sequence L_f of order n. Finally, in Section 6, it is noted that $X^2 - X - 1$ is related to $s_n(X)$ by changing the number of mean proportionals b between a and $a + b$ from one to $n - 1$.

Some of the third order recursion sequences having characteristic polynomial $X^3 - X - 1$ have received quite a bit of attention in recent years. See for example [1, 5, 19, 21, 27, 28, 30, 31] and the references listed there. The polynomials $X^n - X - 1$, $n \geq 4$ or some of their associated recursion sequences have also received some attention recently [9, 19, 20, 24, 26]. We were led to consider the family of polynomials $\{s_n(X) = X^n - X - 1 \mid n \geq 2\}$ and their discriminants in [24], by questions which arose in [23]. To answer these motivating questions in [23] requires computing resultants of pairs of Selmer polynomials. Some results on resultants involving cyclotomic polynomials are given in [3, 4, 8] and [15]. However, the only results we are aware of where the resultants $\mathrm{Res}(f_m, f_n)$ have been computed for all pairs f_m, f_n in families $\{f_n(X) \mid n \geq 1\}$ not involving cyclotomic polynomials are in [10].

2. The Resultants of Pairs of Selmer Polynomials

Let R be an integral domain and let f, $g \in R[X]$. If

$$f = a_0X^m + a_1X^{m-1} + \cdots + a_m = a_0\prod_{i=1}^{m}(X - \alpha_i) \text{ and} \tag{2.1}$$

$$g = b_0X^n + b_1X^{n-1} + \cdots + b_n = b_0\prod_{j=1}^{n}(X - \beta_j), \tag{2.2}$$

then the **resultant** $\text{Res}(f, g)$ of f and g is often defined as the determinant of the $(n+m)$ by $(n+m)$ **Sylvester matrix**

$$\begin{bmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_m & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_m & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_m & 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_m & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_m \\ b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 & \cdot & \cdot & 0 \\ 0 & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & b_{n-1} & b_n & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & b_0 & b_1 & \cdot & \cdot & \cdot & b_{n-1} & b_n \end{bmatrix}.$$

We also have the following formulas [18, pp. 135-138]:

$$\text{Res}(f, g) = a_0^n b_0^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i - \beta_j) \tag{2.3}$$

$$= a_0^n \prod_{i=1}^{m} g(\alpha_i) \tag{2.4}$$

$$= (-1)^{mn} b_0^m \prod_{j=1}^{n} f(\beta_j). \tag{2.5}$$

Thus, if f is monic, we have

$$\text{Res}(f, g) = \prod_{i=1}^{m} g(\alpha_i). \tag{2.6}$$

It follows that $\text{Res}(f, g) \in R$, $\text{Res}(f, g) = fh_1 + gh_2$ for some $h_i \in R[X]$ and that $\text{Res}(f, g) = 0$ if and only if f and g have a common root in some field containing R. The following lemma connects the computation of the resultants $\text{Res}(s_m, s_{m+k})$ to Mahler measure which has been much studied since it was defined in [19]. For some recent work on Mahler measure see for example [2, 7, 11, 14, 29], and the references listed there.

Lemma 2.1. $\text{Res}(s_m, s_{m+k}) = \text{Res}(s_m, X^k - 1)$ *for* $m \geq 2$ *and* $k \geq 0$.

Proof. From Equation (2.6), if $\alpha_1, \ldots, \alpha_m$ are the roots of s_m, we get

$$\text{Res}(s_m, s_{m+k}) = \prod_{j=1}^{m} s_{m+k}(\alpha_j) = \prod_{j=1}^{m} (\alpha_j^{m+k} - \alpha_j - 1) = \prod_{j=1}^{m} (\alpha_j^k \alpha_j^m - \alpha_j - 1),$$

and since α_j is a root of s_m, this is

$$\prod_{j=1}^{m} (\alpha_j^k(\alpha_j + 1) - \alpha_j - 1) = \prod_{j=1}^{m} (\alpha_j^k - 1)(\alpha_j + 1) = \text{Res}(s_m, (X^k - 1)(X + 1)).$$

But from the property $\text{Res}(f, gh) = \text{Res}(f, g)\ \text{Res}(f, h)$, which is immediate from formula (2.4), this is $\text{Res}(s_m, X^k - 1)\text{Res}(s_m, X + 1)$, and applying formula (2.5) to the second factor, we get $\text{Res}(s_m, X^k - 1))(-1)^m s_m(-1) = \text{Res}(s_m, X^k - 1))(-1)^{2m} = \text{Res}(s_m, X^k - 1)$. □

We obtain the following result on the behavior of $\text{Res}(s_m, s_{m+1})$ which is sufficient to answer a motivating question which arises from [23].

Proposition 2.2. *For any integer $m \geq 2$, we have*
(a) $\text{Res}(s_m, s_{m+1}) = (-1)^{m+1}$, *and*
(b) $\lim_{k\to\infty} |\text{Res}(s_m, s_{m+k})| = \infty$.

Proof. (a) From Lemma 2.1 and formula (2.5), $\text{Res}(s_m, s_{m+1}) = \text{Res}(s_m, X-1) = (-1)^m s_m(1) = (-1)^{m+1}$.

(b) If $\alpha_1, \ldots, \alpha_m$ are the roots of s_m, we get from Lemma 2.1 that $|\text{Res}(s_m, s_{m+k})| = |\text{Res}(s_m, X^k - 1)| = \prod_{j=1}^{m} |\alpha_j^k - 1|$, and for one index j, say $j = 1$, we have $\alpha_j > 1$ [26]. Then $|\alpha_1^k - 1|$ goes to infinity with k. For the j with $|\alpha_j| < 1$, the factor $|\alpha_j^k - 1|$ goes to 1 as n goes to infinity. Thus it suffices to show that no α_j has modulus 1. Suppose some $\alpha_j = \alpha$ has modulus 1. Then $1 = |\alpha^m| = |\alpha + 1|$. But the only points α on the unit circle such that $\alpha + 1$ is also on the unit circle are $\alpha = e^{\pm\frac{2\pi i}{3}}$. But since s_m is irreducible over $\mathbb{Q}$ [26, Theorem 1], if some α_j has modulus 1, this would imply that s_m is the minimal polynomial of $\alpha = e^{\pm\frac{2\pi i}{3}}$, which is clearly not the case. (Alternately, one could just show directly that $\alpha = e^{\pm\frac{2\pi i}{3}}$ are not roots of s_m.) □

3. Formulas for $\text{Res}(s_2, s_n)$ in Terms of Fibonacci and Lucas Sequences

For small values of m and n it is easy to compute $\text{Res}(s_m, s_n)$ directly from the definition in terms of the Sylvester matrix. However, this matrix increases with both m and n. So to compute $\text{Res}(s_2, s_n)$ for all n it is useful to convert equation (2.6) into a form that involves only two-by-two matrices of integers. For this we can use the following result which is often called *the Spectral Mapping Theorem*. See [12] for example. If R is a commutative ring and M is a matrix with entries in R, we denote the characteristic polynomial of M by $P_M(X)$.

Theorem 3.1. [18, Theorem 14, p. 404] *If R is a commutative ring, $g \in R[X]$, M is a matrix with entries in R and $P_M(X) = \prod_{i=1}^{m}(X - \alpha_i)$, then $P_{g(M)}(X) = \prod_{i=1}^{m}(X - g(\alpha_i))$. In particular,* $\det(g(M)) = \prod_{i=1}^{m} g(\alpha_i)$.

To apply this to equation (2.6) requires choosing a matrix M with $f = P_M(X) = \prod_{i=1}^{m}(X - \alpha_i)$. For $f = X^m + a_{m-1}X^{m-1} + \cdots + a_0 \in R[X]$, we take the companion matrix of f to be

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & & \dots & & & -a_{m-2} & -a_{m-1} \end{bmatrix}.$$

Some other matrices that are often called "the companion matrix of f" are the transpose $C(f)^T$ of $C(f)$, and the matrices obtained by reversing the rows and columns of $C(f)$ or $C(f)^T$. An easy induction, expanding on the first column, shows that the characteristic polynomial of $C(f)$ is f. Thus by Theorem 3.1 and equation (2.6), we have the following **companion matrix form of the resultant**. (See [6, Theorem 1.3, p. 15], [12, Theorem 5.1].)

$$\operatorname{Res}(f, g) = \prod_{i=1}^{m} g(\alpha_i) = \det(g(C(f))). \tag{3.1}$$

The companion matrix C_m of $X^m - X - 1$ is

$$C(s_m) = C_m = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\operatorname{Res}(s_m, s_n) = \det(s_n(C_m)) = |(C_m)^n - C_m - I_m|$.

For the degree 2 case, we may use the well-known fact that

$$(C_2)^n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}, \tag{3.2}$$

where f_n is the nth Fibonacci number. Also recall the identity

$$f_{n+1}f_{n-1} = f_n^2 + (-1)^n, \tag{3.3}$$

which can be seen by taking the determinant of both sides of equation (3.2).

Proposition 3.2. $\operatorname{Res}(s_2, s_{2+k}) = (-1)^k - f_{k+1} - f_{k-1} + 1$ *for* $k \geq 1$.

Proof. Using Lemma 2.1 and the companion matrix form of the resultant, we get

$$\operatorname{Res}(s_2, s_{2+k}) = \operatorname{Res}(s_2, X^k - 1) = \left| \begin{bmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{bmatrix} - I_2 \right|$$

$$= \left| \begin{bmatrix} f_{k-1} - 1 & f_k \\ f_k & f_{k+1} - 1 \end{bmatrix} \right| = (f_{k+1} - 1)(f_{k-1} - 1) - f_k^2 = (-1)^k - f_{k+1} - f_{k-1} + 1,$$

by using the identity (3.3). □

Recall that **Binet's Formula** says that if α and β denote the positive and negative roots, respectively of $s_2(X) = X^2 - X - 1$, then

$$f_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{3.4}$$

Remark 3.3. It is possible to also derive the formula in Proposition 3.2 from equation (2.6) and formula (3.4). Indeed since $\alpha\beta = -1$, we have from Lemma 2.1 and formula (2.6), $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2, X^k - 1) = (\alpha^k - 1)(\beta^k - 1) = (\alpha\beta)^k - \alpha^k - \beta^k + 1 = (-1)^k - \alpha^k - \beta^k + 1 = (-1)^k - \left(\frac{(\alpha^k+\beta^k)(\alpha-\beta)}{(\alpha-\beta)}\right) + 1$, and using equation(3.4), gives $(-1)^k - \left(\frac{(\alpha^{k+1}-\beta^{k+1})+(\alpha^{k-1}-\beta^{k-1})}{(\alpha-\beta)}\right) + 1 = (-1)^k - f_{k+1} - f_{k-1} + 1$.

Recall that the **Lucas sequence** $\{L(n) \mid n \in \mathbb{Z}\}$ is defined by $L(1) = 1$, $L(2) = 3$ and $L(n+2) = L(n+1) + L(n)$. The counterpart to the Binet Formula for the Lucas sequence is

$$L(n) = \alpha^n + \beta^n. \tag{3.5}$$

Since this is simpler than the Binet Formula, the computation in Remark 3.3 shows that it is simpler to express $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2, X^k - 1)$ in terms of the Lucas sequence. Indeed the computation in Remark 3.3 gives the following variation of Proposition 3.2.

Proposition 3.4. $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2, X^k - 1) = (-1)^k - L(k) + 1$ *for* $k \geq 1$.

For later reference, we list some values for $L(k)$ and $\mathrm{Res}(s_2, s_{2+k})$.

$$\begin{bmatrix} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\dots \\ L(k) & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47\dots \\ \mathrm{Res}(s_2, s_{2+k}) & 0 & -1 & -1 & -4 & -5 & -11 & -16 & -29 & -45\dots \end{bmatrix}. \tag{3.6}$$

So we see that computing $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2, X^k - 1)$ is about the same as computing $L(k)$. Recall that a **linear recurrence sequence** in a field K is a sequence $a = \{a(i) \mid i \in \mathbb{N}_0\}$ in K satisfying a linear recurrence relation

$$a(i+n) = r_1 a(i+n-1) + \cdots + r_{n-1} a(i+1) + r_n a(i) \tag{3.7}$$

with $r_i \in K$. Then a is said to have **order** n and the polynomial

$$f(X) = X^n - r_1 X^{n-1} - \cdots - r_{n-1} X - r_n \tag{3.8}$$

is called the **characteristic polynomial of** a. The recurrence sequence a is uniquely determined by its initial values $a(1), a(2), \dots, a(n)$ (or any n consecutive values) and its characteristic polynomial. So the Selmer polynomial $s_2(X)$ is the characteristic polynomial of the Fibonacci and Lucas sequences. If f has distinct roots $\alpha_1, \dots, \alpha_n$ then for each $i \in \{1, 2, \dots, n\}$, the sequence of powers $\alpha_i^0, \alpha_i^1, \alpha_i^2, \alpha_i^3, \dots$ is a linear recurrence sequence with characteristic polynomial $f(X)$, and any other linear recurrence sequence b with characteristic polynomial $f(X)$ is of the form $b(i) = b_1\alpha_1^i + b_2\alpha_2^i + \cdots + b_n\alpha_n^i$ for unique b_i in a splitting field of K. This is because we have n initial conditions, and the vectors $(1, \alpha_1, \alpha_1^2, \dots, \alpha_1^{n-1})$, $(1, \alpha_2, \alpha_2^2, \dots, \alpha_2^{n-1})$, $\dots$, $(1, \alpha_n, \alpha_n^2, \dots, \alpha_n^{n-1})$ are linearly independent since the corresponding Vandermonde determinant is not zero. Then the Binet Formula mentioned earlier for the Fibonacci sequence is a special case of the formula $b(i) = b_1\alpha_1^i + b_2\alpha_2^i + \cdots + b_n\alpha_n^i$.

Returning to the sequence $\mathrm{Res}(s_2, s_{2+k}) = \mathrm{Res}(s_2, X^k - 1) = (-1)^k - L(k) + 1$, it follows, by comparing the above equations (3.7) and (3.8), that the recursion sequences $A(n) = 1$, $B(n) = (-1)^n$ and $C(n) = -L(n)$ have characteristic polynomials $(X - 1)$, $(X + 1)$ and $(X^2 - X - 1)$ with initial conditions $A(0) = 1$, $B(0) = 1$, and $(C(0), C(1)) = (2, 1)$, respectively.

But by [13, Theorem 1.1, page 3], the least common multiple of the characteristic polynomials of the recursion sequences $A(n)$, $B(n)$, and $C(n)$ is a characteristic polynomial for their sum $\mathrm{Res}(s_2, s_{2+k})$. In this case, the least common multiple of $(X-1)$, $(X+1)$, and (X^2-X-1) is their product. So $\mathrm{Res}(s_2, s_{2+k})$ has characteristic polynomial $X^4 - X^3 - 2X^2 + X + 1 = (X-1)(X+1)(X^2-X-1)$. Thus, if we write $R_2(k)$ for $\mathrm{Res}(s_2, s_{2+k})$, it follows that R_2 satisfies the recursion rule

$$R_2(k+4) = R_2(k+3) + 2R_2(k+2) - R_2(k+1) - R_2(k)$$

with initial conditions $R_2(0) = 0$, $R_2(1) = -1$, $R_2(2) = -1$, $R_2(3) = -4$ as given in the last line of the table (3.6), and characteristic polynomial $X^4 - X^3 - 2X^2 + X + 1 = (X-1)(X+1)(X^2 - X - 1)$. Indeed as in table (3.6), we have $R_2(4) = R_2(3) + 2R_2(2) - R_2(1) - R_2(0) = -4 + 2(-1) - (-1) - 0 = -5$, $R_2(5) = R_2(4) + 2R_2(3) - R_2(2) - R_2(1) = -5 + 2(-4) - (-1) - (-1) = -11$ and so on.

4. Formulas for $\mathrm{Res}(s_3, s_{3+k})$ in Terms of the Perrin Sequence

From the experience of computing $\mathrm{Res}(s_2, s_{2+k})$ in Section 3, it appears that instead of trying to maintain the analogy with equation (3.2), perhaps we should choose the initial conditions to simplify the Binet Formula. Following Lucas' lead, we let α_1, α_2, α_3 be the roots of $s_3(X) = X^3 - X - 1$ and let

- $L_3(1) = \alpha_1 + \alpha_2 + \alpha_3 = 0$,
- $L_3(2) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 2$, and
- $L_3(3) = \alpha_1^3 + \alpha_2^3 + \alpha_3^3 = 3$.

Indeed comparing coefficients of X^i in the equation $X^3 - X - 1 = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$ gives

$$0 = \alpha_1 + \alpha_2 + \alpha_3, \quad -1 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad \text{and} \quad 1 = \alpha_1\alpha_2\alpha_3. \tag{4.1}$$

This gives the first equation above and the second follows from $(\alpha_1 + \alpha_2 + \alpha_3)^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$. The third follows since $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = (\alpha_1 + 1) + (\alpha_2 + 1) + (\alpha_3 + 1)$. Thus we have for all $n \geq 0$,

$$L_3(n) = \alpha_1^n + \alpha_2^n + \alpha_3^n. \tag{4.2}$$

We list some values of $L_3(n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 ⋯
$L_3(n)$	3	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68	90	119	158 ⋯

From Lemma 2.1 and formula (2.6) we get

$$\begin{aligned}
\mathrm{Res}(s_3, s_{3+k}) &= \mathrm{Res}(s_3, X^k - 1) = R_3(k) \\
&= \prod_{i=1}^{3} ({\alpha_i}^k - 1) = (\alpha_1\alpha_2\alpha_3)^k - (\alpha_1\alpha_2)^k - (\alpha_1\alpha_3)^k - (\alpha_2\alpha_3)^k + {\alpha_1}^k + {\alpha_2}^k + {\alpha_3}^k - 1 \\
&= -(\alpha_1\alpha_2)^k - (\alpha_1\alpha_3)^k - (\alpha_2\alpha_3)^k + {\alpha_1}^k + {\alpha_2}^k + {\alpha_3}^k.
\end{aligned}$$

Thus, $R_3(k) = A(k) + B(k)$ where $A(k) = {\alpha_1}^k + {\alpha_2}^k + {\alpha_3}^k = L_3(k)$ is the degree three version of the Lucas sequence and $B(k) = -(\alpha_1\alpha_2)^k - (\alpha_1\alpha_3)^k - (\alpha_2\alpha_3)^k$ is another Lucas type linear recurrence sequence with characteristic polynomial $g(X) = (X - \alpha_1\alpha_2)(X - \alpha_1\alpha_3)(X - \alpha_2\alpha_3) = \prod_{i=1}^{3}(X - \frac{1}{\alpha_i})$. We know that the product of the characteristic polynomial s_3 for $A(k)$ and the characteristic polynomial g for $B(k)$ is a characteristic polynomial for $R_3(k)$. Using equations (4.1), it follows that

$$g(X) = X^3 + X^2 - 1.$$

(Observe that $g(X) = -X^3 s_3(1/X) = -X^3((\frac{1}{X})^3 - \frac{1}{X} - 1)$, the negative of the reciprocal polynomial of $s_3(X)$.) So a characteristic polynomial for $R_3(k)$ is

$$(X^3 - X - 1)(X^3 + X^2 - 1) = X^6 + X^5 - X^4 - 3X^3 - X^2 + X + 1.$$

So $R_3(k)$ satisfies the recurrence formula

$$R_3(k) = -R_3(k-1) + R_3(k-2) + 3R_3(k-3) + R_3(k-4) - R_3(k-5) - R_3(k-6). \quad (4.3)$$

Therefore to determine the sequence $R_3(k)$ it remains only to determine 6 consecutive values of $R_3(k)$. If in the above formula

$$R_k(3) = ({\alpha_1}^k - 1)({\alpha_2}^k - 1)({\alpha_3}^k - 1) = -(\alpha_1\alpha_2)^k - (\alpha_1\alpha_3)^k - (\alpha_2\alpha_3)^k + {\alpha_1}^k + {\alpha_2}^k + {\alpha_3}^k,$$

we let ${\alpha_i}^k = x_i$, then the above states

$$\prod_{i=1}^{3}(x_i - 1) = -\sigma_2(x_1, x_2, x_3) + \sigma_1(x_1, x_2, x_3)$$

where $\sigma_i(x_1, x_2, x_3)$ is the ith elementary symmetric function in the $x_i = {\alpha_i}^k$. But Newton's Formulas allow us to express the elementary symmetric functions in terms of the power functions $p_i(x_1, x_2, x_3) = {x_1}^i + {x_2}^i + {x_1}^i$. If we use the form of Newton's Formulas in terms of determinants [22, page 79], this gives the following counterpart to Proposition 3.4.

Proposition 4.1. $\operatorname{Res}(s_3, s_{3+k}) = -\frac{1}{2}\begin{vmatrix} L_3(k) & 1 \\ L_3(2k) & L_3(k) \end{vmatrix} + L_3(k)$ *for* $k \geq 1$.

Proof. Using ${\alpha_i}^k = x_i$, Newton's Formulas gives

$$\begin{aligned} \operatorname{Res}(s_3, X^k - 1) &= \prod_{i=1}^{3}(x_i - 1) \\ &= -\frac{1}{2}\begin{vmatrix} p_1(x) & 1 \\ p_2(x) & p_1(x) \end{vmatrix} + p_1(x) \\ &= -\frac{1}{2}\begin{vmatrix} L_3(k) & 1 \\ L_3(2k) & L_3(k) \end{vmatrix} + L_3(k) \quad \text{for } k \geq 1, \end{aligned}$$

where the last equality is obtained by replacing x_i with α_i^k and using the definition of the generalized Lucas sequence $L_3(k)$. □

Thus we have the following values for $R_3(k) = \operatorname{Res}(s_3, s_k)$, which can be computed either directly from Proposition 4.1 or by using Proposition 4.1 to compute six initial values, and then using equation (4.3).

$$\begin{bmatrix} k & \cdots & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ \operatorname{Res}(s_3, s_{3+k}) & \cdots & 0 & 1 & 1 & 1 & 5 & 1 & 7 & 8 & 5 & 19 & \cdots \end{bmatrix}.$$

5. The General Case

Let D be an integral domain with quotient field K. In this section we switch from $\text{Res}(s_m(X), X^k - 1)$ to $\text{Res}(f(X), X^k - 1)$ where $f(X) \in D[X]$. For simplicity we assume that D has characteristic zero, $f(X)$ is a monic separable polynomial, $f(0) \neq 0$ and f has no root of unity as a zero. If $\alpha_1, \alpha_2, \dots, \alpha_n$ are the (distinct) roots of f, we define the **Lucas sequence L_f with characteristic polynomial** $f(X)$ by $L_f(k) = \Sigma_{i=1}^n {\alpha_i}^k$ [9].

Let X and $t_1, \dots, t_n$ be independent indeterminates and define $\overline{f}(X) \in \mathbb{Z}[t_1, \dots, t_n, X]$ by

$$\overline{f}(X) = \prod_{i=1}^{n}(X - t_i) = X^n - \sigma_1(t)X^{n-1} + \sigma_2(t)X^{n-2} - \sigma_3(t)X^{n-3} + \cdots + (-1)^n\sigma_n(t).$$

So $\sigma_1, \sigma_2, \dots, \sigma_n$, are the elementary symmetric polynomials in $t_1, \dots, t_n$. If we set $t_i = {\alpha_i}^k$ in $\sigma_j(t_1, t_2, \dots, t_n) = \sigma_j(t)$, then since $f(X) \in D[X]$ and $\sigma_j({\alpha_1}^k, {\alpha_2}^k, \dots, {\alpha_n}^k) = \sigma_j(\alpha^k)$ is symmetric in the α_i, it follows from a classical theorem on symmetric polynomials, for example [18, page 133], that $\sigma_j(\alpha^k) \in D$ for $j = 1, \dots, n$ and each $k \in \mathbb{N}$. In particular, $\Sigma_{i=1}^n {\alpha_i}^k = L_f(k) \in D$ for each $k \in \mathbb{N}$.

We have $\overline{f}(1) = \prod_{i=1}^n(1 - t_i) = 1 - \sigma_1(t) + \sigma_2(t) - \sigma_3(t) + \cdots + (-1)^n\sigma_n(t)$, and thus

$$\begin{aligned}\prod_{i=1}^{n}(t_i - 1) &= (-1)^n\overline{f}(1)\\ &= (-1)^n + (-1)^{n-1}\sigma_1(t) + (-1)^{n-2}\sigma_2(t) + (-1)^{n-3}\sigma_3(t) + \cdots + \sigma_n(t).\end{aligned}$$

By Newton's Formulas for writing the $\sigma_j(t)$ in terms of the power functions $p_i(t) = {t_1}^i + {t_2}^i + \cdots + {t_n}^i$ [22, page 79], we have

$$\sigma_m(t) = \frac{1}{m!}\begin{vmatrix} p_1(t) & 1 & 0 & \dots & 0 \\ p_2(t) & p_1(t) & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m-1}(t) & p_{m-2}(t) & p_{m-3}(t) & \dots & m-1 \\ p_m(t) & p_{m-1}(t) & p_{m-2}(t) & \dots & p_1(t) \end{vmatrix}.$$

Substituting α_i^k for t_i gives

$$\begin{aligned}\text{Res}(f, X^k - 1) &= \prod_{i=1}^{n}(\alpha_i^k - 1)\\ &= (-1)^n + (-1)^{n-1}\sigma_1(\alpha^k) + (-1)^{n-2}\sigma_2(\alpha^k) + (-1)^{n-3}\sigma_3(\alpha^k) + \cdots + \sigma_n(\alpha^k),\end{aligned}$$

where

$$\sigma_m(\alpha^k) = \frac{1}{m!}\begin{vmatrix} p_1(\alpha^k) & 1 & 0 & \dots & 0 \\ p_2(\alpha^k) & p_1(\alpha^k) & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m-1}(\alpha^k) & p_{m-2}(\alpha^k) & p_{m-3}(\alpha^k) & \dots & m-1 \\ p_m(\alpha^k) & p_{m-1}(\alpha^k) & p_{m-2}(\alpha^k) & \dots & p_1(\alpha^k) \end{vmatrix}.$$

Using $L_f(jk) = p_j(\alpha^k) = \alpha_1^{jk} + \cdots + \alpha_n^{jk}$, we get

$$\sigma_m(\alpha^k) = \frac{1}{m!}\begin{vmatrix} L_f(k) & 1 & 0 & \dots & 0 \\ L_f(2k) & L_f(k) & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_f((m-1)k) & L_f((m-2)k) & L_f((m-3)k) & \dots & m-1 \\ L_f(mk) & L_f((m-1)k) & L_f((m-2)k) & \dots & L_f(k) \end{vmatrix}. \tag{5.1}$$

Theorem 5.1. *If $f \in D[X]$ is as above, then $\operatorname{Res}(f, X^k - 1) = (-1)^n + (-1)^{n-1}\sigma_1(\alpha^k) + (-1)^{n-2}\sigma_2(\alpha^k) + (-1)^{n-3}\sigma_3(\alpha^k) + \cdots + \sigma_n(\alpha^k)$, where $\sigma_m(\alpha^k)$ is as in equation (5.1).*

The values of $\operatorname{Res}(f, X^k - 1)$ can be computed for each $k \in \mathbb{N}_0$ directly from the formula in Theorem 5.1 after first finding the initial n values of L_f. To compute $L_f(m)$ for $m = 0, 1, \dots, n-1$, we put $t_i = \alpha_i$ in the Newton Formula, [22, page 79],

$$p_m(t) = \begin{vmatrix} \sigma_1(t) & 1 & 0 & \dots & 0 \\ 2\sigma_2(t) & \sigma_1(t) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-1)\sigma_{m-1}(t) & \sigma_{m-2}(t) & \sigma_{m-3}(t) & \dots & 1 \\ m\sigma_m(t) & \sigma_{m-1}(t) & \sigma_{m-2}(t) & \dots & \sigma_1(t) \end{vmatrix}.$$

In particular for $s_4(X) = X^4 - X - 1 = X^4 - \sigma_1(\alpha)X^3 + \sigma_2(\alpha)X^2 - \sigma_3(\alpha)X + \sigma_4(\alpha)$, we have $L_{s_4}(0) = L_4(0) = p_0(\alpha) = 4$, $L_4(1) = p_1(\alpha) = \sigma_1(\alpha) = 0$,

$$L_4(2) = p_2(\alpha) = \begin{vmatrix} \sigma_1(\alpha) & 1 \\ 2\sigma_2(\alpha) & \sigma_1(\alpha) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0.$$

$$L_4(3) = p_3(\alpha) = \begin{vmatrix} \sigma_1(\alpha) & 1 & 0 \\ 2\sigma_2(\alpha) & \sigma_1(\alpha) & 1 \\ 3\sigma_3(\alpha) & \sigma_2(\alpha) & \sigma_1(\alpha) \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} = 3.$$

We recall the following result on linear recursion sequences.

Theorem 5.2. *Let $\{A_1(n) \mid n \in \mathbb{N}_0\}$ and $\{A_2(n) \mid n \in \mathbb{N}_0\}$ be linear recurrence sequences of orders m_1 and m_2, respectively in a field K.*

(1) *Then $\{B(n) = A_1(n) + A_2(n) \mid n \in \mathbb{N}_0\}$ and $\{C(n) = A_1(n)A_2(n) \mid n \in \mathbb{N}_0\}$ are linear recurrence sequences in K of order at most $m_1 + m_2$ and m_1m_2, respectively* [13, Theorem 4.1].
(2) *If $\{A(n) \mid n \in \mathbb{N}_0\}$ is a linear recurrence sequence of order m in a field K and $k, \ell \in \mathbb{N}_0$, then $\{A(kn + \ell) \mid n \in \mathbb{N}_0\}$ is a linear recurrence sequence of order at most m* [13, Theorem 1.3].

We summarize the above in the following theorem.

Theorem 5.3. *If the monic polynomial $f(X) \in D[X]$ of degree n satisfies the conditions stated at the beginning of this section, then the sequence $\{\operatorname{Res}(f, X^k - 1) \mid k \geq 0\}$ is a linear recurrence sequence in D.*

Proof. If we write $f(X)$ as $X^n - \sum_{i=1}^n a_i X^{n-i}$, it is clear that for each root α_j of $f(X)$, the sequence $\{\alpha_j^k \mid k \geq 0\}$ satisfies the recursion rule $\alpha_j^{k+n} = \sum_{i=1}^n a_i \alpha_j^{k+n-i}$ of order n. Thus the Lucas sequence $\{L_f(k) = \alpha_1^k + \alpha_2^k + \cdots + \alpha_n^k \mid k \geq 0\}$ satisfies this recursion rule, and as noted above, $L_f(k) \in D$ for each k. By Theorem 5.2 (2), for each fixed $j \in \mathbb{N}$, the sequence $\{L_f(jk) \mid k \geq 0\}$ is a recurrence sequence. Thus by Theorem 5.1, for each fixed $k \geq 0$, the kth term $\operatorname{Res}(f, X^k - 1)$ of $\{\operatorname{Res}(f, X^k - 1) \mid k \geq 0\}$ is a sum of products of the kth terms of various recurrence sequences, then by Theorem 5.2 (1), $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 0\}$ is also a linear recurrence sequence. $\square$

An alternate way of thinking about the formula

$$\mathrm{Res}(f, X^k - 1) = (-1)^n + (-1)^{n-1}\sigma_1(\alpha^k) + (-1)^{n-2}\sigma_2(\alpha^k) + (-1)^{n-3}\sigma_3(\alpha^k) + \cdots + \sigma_n(\alpha^k)$$

in Theorem 5.1 is to consider $\sigma_j(\alpha^k) = \sigma_j(\alpha_1^k, \ldots, \alpha_n^k) = \sum_{i_1<\cdots<i_j}(\alpha_{i_1}\cdots\alpha_{i_j})^k$ as the Lucas sequence L_{f_j} with characteristic polynomial

$$f_j(X) = \prod_{i_1<\cdots<i_j}(X - \alpha_{i_1}\cdots\alpha_{i_j}) \text{ of degree } \binom{n}{j}.$$

We note that any permutation τ in the symmetric group S_n induces an automorphism τ' of $K(t_1, \ldots, t_n)$ and we let $\overline{\tau}$ be the induced automorphism of $K(t_1, \ldots, t_n)[X]$ obtained by applying τ' to the coefficients of any $g \in K(t_1, \ldots, t_n)[X]$. Then $\overline{\tau}$ fixes $\overline{f}_j(X) = \prod_{i_1<\ldots<i_j}(X - t_{i_1}\cdots t_{i_j})$ and it follows that the coefficients of $\overline{f}_j(X)$ are in $D[\sigma_1(t), \ldots, \sigma_n(t)]$ and hence the coefficients of $f_j(X) = \prod_{i_1<\ldots<i_j}(X - \alpha_{i_1}\cdots\alpha_{i_j})$ are in $D[\sigma_1(\alpha), \ldots, \sigma_n(\alpha)] = D$. So the coefficients of $f_j(X)$ and the terms of the Lucas sequence $L_{f_j}(k) = \sigma_j(\alpha^k) = \sigma_j(\alpha_1^k, \ldots, \alpha_n^k) = \sum_{i_1<\cdots<i_j}(\alpha_{i_1}\cdots\alpha_{i_j})^k$ with characteristic polynomial $f_j(X)$ are in D. So any multiple of the least common multiple $M(X) \in D[X]$ of the $f_j(X)$ is a characteristic polynomial for the sequence $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 0\}$. Thus, $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 0\}$ has degree $\leq \sum_{i=0}^{n}\binom{n}{j} = 2^n$. This description of the recurrence sequence $\{\mathrm{Res}(f(X), X^k - 1) \mid k \in \mathbb{N}_0\}$ is given in [19, Theorem 13]. In the case of $f = s_4(X) = X^4 - X - 1$, a computation similar to the one in Section 4 for the case $f = s_3(X)$ gives $f_0 = X - 1$, $f_1 = s_4(X)$, $f_2 = X^6 - X^4 + X^3 - X^2 - 1$, $f_3 = X^4 - X^3 - 1$, $f_4 = X + 1$. After making this computation by hand we noticed that the corresponding formulas are given in [19, Theorem 13] for arbitrary monic $f \in \mathbb{Z}[X]$ of degree ≤ 5. We expect that closed form formulas for the $f_i(X)$, similar to Newton's Formulas, are known for any monic $f \in \mathbb{Z}[X]$ of any degree, but do not know of a reference. However, if for example the f_i are pairwise relatively prime, so that the least common multiple of these polynomials is their product, then $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 1\}$ is a linear recurrence sequence of order 2^n and hence, to describe the sequence $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 1\}$ from the polynomials f_i, one must compute $\mathrm{Res}(f, X^k - 1)$ for 2^n initial (or consecutive) values of k whereas, in using the formula in Theorem 5.1, the sequence $\{\mathrm{Res}(f, X^k - 1) \mid k \geq 0\}$ is determined from only n consecutive values of k (in the sense that after the simple computation of $L_f(k)$ for n consecutive values of k, we can take $L_f(k)$ and hence $\mathrm{Res}(f, X^k - 1)$ as known for all k).

6. Historical Note

There are many different recurrence sequences which are called generalized Fibonacci sequences and several polynomials which are considered to be generalizations of the polynomial $X^2 - X - 1$, for example $X^n - X^{n-1} - 1$ or $X^n - X^{n-1} - \cdots - X - 1$. The object of this section is to point out how the unique positive root α_n of $X^n - X - 1$ could be considered to be the degree n version of the golden ratio.

It is well-known that the Greek mathematician Hippocrates of Chios (460-380 B.C.E.) showed that the problem of doubling the cube can be converted to the problem of constructing two mean proportionals between a length a and its double $2a$. He may have surmised this from the fact that given a square with sides of length a, constructing a square with side length b so that $b^2 = 2a^2$, amounts to constructing a mean proportional b between a and $2a$. This suggests that for the original three dimensional problem, perhaps one should consider finding two mean proportionals b and c between a and $2a$. That is find b and c such that $\frac{b}{a} = \frac{c}{b} = \frac{2a}{c}$. Indeed if we have b and c, then $b^2 = ac$ and $c^2 = 2ab$ imply $b^4 = a^2c^2 = a^2(2ab) = 2a^3b$ and thus $b^3 = 2a^3$. (For more on Hippocrates' work on doubling the cube, including how he may have reduced the problem to the construction of two mean proportionals, see for example [16] and [25].)

We are not aware if the Greek mathematicians ever considered a three dimensional analogue to the golden ratio by seeking lengths a, b and c so that $\frac{b}{a} = \frac{c}{b} = \frac{a+b}{c}$. That is, b and c would give two mean proportionals between a and $a+b$. However, according to Richard Padovan's book [21, page 85], in considering questions in architectural design, the French Benedictine monk-architect Dom Hans van der Laan was led, around 1928, to consider the result of dividing a segment AB into three segments of lengths a, b, and c such that

$$\frac{b}{a} = \frac{c}{b} = \frac{a+b}{c} = \frac{b+c}{a+b} = \frac{a+b+c}{b+c}.$$

Thus, in this case the common ratio $r = \frac{b}{a}$ gives four mean proportionals, b, c, $a+b$ and $b+c$ between the length a and the length $a+b+c$ of AB. It is immediate however that, for example, the last 2 equalities above are consequences of the first two. So, we are really just choosing a, b and c so that $\frac{b}{a} = \frac{c}{b} = \frac{a+b}{c}$. It is immediate that the equalities $\frac{b}{a} = \frac{c}{b} = \frac{a+b}{c}$ imply that this common ratio $r = \frac{b}{a}$ is a root of the polynomial $X^3 - X - 1$. This unique positive root of $X^3 - X - 1$ was called the **plastic number** by Dom Hans van der Laan. It has many properties which are similar to properties of the golden ratio [30].

In order to pass from 3 to n, suppose we divide a line segment AB into n segments of lengths $a_1, a_2, \ldots, a_n$ such that

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_{n-1}}{a_{n-2}} = \frac{a_n}{a_{n-1}} = \frac{a_1 + a_2}{a_n}.$$

Then $a_{i+1} = ra_i$ for $i = 1, 2, \ldots, n-1$, and thus $a_i = r^{i-1}a_1$ for $i = 1, 2, \ldots, n$. Then $r = \frac{a_1+a_2}{a_n} = \frac{1+r}{r^{n-1}}$ implies $r^n = r + 1$. Also it follows that

$$\begin{aligned}\frac{a_1+a_2}{a_n} &= \frac{a_2+a_3}{a_1+a_2} = \frac{a_3+a_4}{a_2+a_3} = \cdots = \frac{a_{n-1}+a_n}{a_{n-2}+a_{n-1}} = \frac{a_1+a_2+a_n}{a_{n-1}+a_n} \\ &= \frac{a_2+a_3+a_4}{a_1+a_2+a_3} = \frac{a_3+a_4+a_5}{a_2+a_3+a_4} = \cdots = \frac{a_{n-2}+a_{n-1}+a_n}{a_{n-3}+a_{n-2}+a_{n-1}} = \frac{a_1+a_2+a_{n-1}+a_n}{a_{n-2}+a_{n-1}+a_n} \\ &= \frac{a_2+a_3+a_4+a_5}{a_1+a_2+a_3+a_4} = \cdots = \frac{a_2+a_3+a_4+\cdots+a_n}{a_1+a_2+a_3+\cdots+a_{n-1}} = \frac{a_1+a_2+a_3+\cdots+a_n}{a_2+a_3+\cdots+a_n}.\end{aligned}$$

The common ratio r for this choice of n is the unique positive root of the polynomial $s_n(X) = X^n - X - 1$ [26], which in [24] is called the nth **Selmer polynomial**, after E. S. Selmer who showed that these polynomials $s_n(X)$ have interesting properties including being irreducible over $\mathbb{Q}$ for all n [26]. The equation $\frac{a_n}{a_{n-1}} = \frac{a_1+a_2}{a_n}$ also translates to $\frac{r^{n-1}a_1}{r^{n-2}a_1} = \frac{a_1+a_2}{r^{n-2}a_2}$. So $r^{n-1} = \frac{a_1+a_2}{a_2}$, which yields the following equation which was given in [17]

$$\left(\frac{a_2}{a_1}\right)^{n-1} = \frac{a_1+a_2}{a_2}. \tag{6.1}$$

The equation (6.1) gives another way of seeing that $r = \frac{a_2}{a_1}$ satisfies $s_n(X) = X^n - X - 1$. It was pointed out by V. Krčadinac in [17] that putting $t = r^{n-1} = \frac{1}{r} + 1$ in equation (6.1), solving $t = \frac{1}{r} + 1$ for r and substituting into $s_n(X)$ yields a related polynomial $X(X-1)^{n-1} -$

$1 = k_n(X)$ which has t as its unique positive root. Further, the recurrence sequence with characteristic polynomial $k_n(X)$ and having its n initial values all equal to one has some interesting properties in common with the Fibonacci sequence. Observe that if we denote the **reciprocal polynomial** $X^n f(X^{-1})$ of a degree n polynomial $f(X)$ by $\overline{f}(X)$, then the substitution $X = Y + 1$ transforms $k_n(X)$ into $-\overline{s_n}(Y)$. In particular $k_n(X)$ is irreducible by Selmer's result that $s_n(X)$ is irreducible.

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