ON THE CHARACTERIZATION OF PERIODIC COMPLEX HORADAM SEQUENCES

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ABSTRACT. Horadam sequences are second-order linear recurrence sequences which depend on a family of four parameters (two in the defining recursion itself, and two initial values). In this article we find necessary and sufficient conditions for the periodicity of complex Horadam sequences, under general initial values, characterizing sequence behavior for degenerate and non-degenerate characteristic solution types. Inner and outer boundaries for regions containing periodic orbits are also determined.

1. INTRODUCTION

Let a, b, p and q be complex numbers, and denote by $\{w_n(a, b; p, q)\}_{n=0}^{\infty}$ the sequence defined by the recurrence

$$w_{n+2} - pw_{n+1} + qw_n = 0, \quad w_0 = a, w_1 = b. \tag{1.1}$$

Following the work of A. F. Horadam—who initiated the investigation of this general recursion in two seminal 1965 papers [5] and [6]—the sequence arising from (1.1) is called a Horadam sequence which contains many well-known sequences as special cases. A historical perspective on results related to Horadam sequences is given in a survey paper by Larcombe *et al.* [8]. Under certain conditions Horadam sequences can be periodic. A comprehensive list of periodic recurrence sequences is detailed in [2, Chapter 3], with an emphasis on sequences defined on finite fields.

In this paper we establish necessary and sufficient conditions for the periodicity of complex Horadam sequences using closed forms for the general term $w_n(a, b; p, q)$. The results are formulated in terms of initial sequence values a, b, and so called generators z_1, z_2 which are solutions of the quadratic characteristic equation

$$z^2 - pz + q = 0, \tag{1.2}$$

associated with the recurrence (1.1) and connect the fundamental characteristic polynomial $z^2 - pz + q$ to the sequence it creates through the relations $z_1 + z_2 = p$, $z_1 z_2 = q$. After some preliminary results Horadam sequences are characterized for degenerate and non-degenerate characteristic root cases, exploring the roles of both initial values and generators in periodic (and non-periodic) sequence behavior. To finish, inner and outer boundaries for regions containing periodic orbits are also determined. This article forms the basis of a systematic examination of Horadam sequence cyclicity which, in the light of [8], opens up a new topic of study in what is a long established field within discrete mathematics.

2. PRELIMINARY RESULTS

In this section we discuss the behavior of a sequence $\{z^n\}_{n=0}^{\infty}$ for an arbitrary value of $z \in \mathbb{C}$. We begin with a corollary of Weyl's criterion in the theory of Diophantine equations [9] which gives, as a consequence, that the set of fractional parts of multiples of an irrational number is equally distributed in $[0,1]$ (see [3, Chapter 2]). An adapted and simplified version of the proof presented in [4] is detailed here, which shows that the fractional part of multiples of an irrational number is dense in $[0,1]$.

Let $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \le x\}$ and $\{x\} = x - \lfloor x \rfloor$ be the (resp.) floor and fractional part of x.

Lemma 2.1. *The set $M = \{\{nx\} \mid n \in \mathbb{N}\}$ is dense in the interval $[0,1]$ for every $x \in \mathbb{R}\backslash\mathbb{Q}$.*

Proof. As x is irrational, the numbers $\{nx\}$ are distinct and so the set M contains infinitely many terms. For $2 \le m \in \mathbb{N}$ the interval $[0,1]$ can be partitioned into m subintervals

$$\left[0, \frac{1}{m}\right], \left[\frac{1}{m}, \frac{2}{m}\right], \ldots, \left[\frac{m-1}{m}, 1\right], \tag{2.1}$$

and we will show that each of these intervals contains at least one term $\{nx\}$.

From the well-known Pigeonhole Principle there is an interval containing at least two of the first $m+1$ terms, say $\{m_1 x\}$ and $\{m_2 x\}$ $(0 \le m_1 < m_2 \le m)$, which satisfy the relations $|\{m_2 x\} - \{m_1 x\}| < \frac{1}{m}$ and $(m_2 - m_1)x = \lfloor m_2 x \rfloor - \lfloor m_1 x \rfloor + \{m_2 x\} - \{m_1 x\}$. If $\{m_1 x\} < \{m_2 x\}$ then $\{(m_2 - m_1)x\} = \{m_2 x\} - \{m_1 x\} < \frac{1}{m}$. The same argument applies when $\{m_1 x\} > \{m_2 x\}$, hence one can write $\{(m_2 - m_1)x\} = 1 + \{m_2 x\} - \{m_1 x\} \in (1 - \frac{1}{m}, 1)$. The presence of $\{m_1 x\}$ and $\{m_2 x\}$ in the same interval of length $1/m$ ensures the existence of at least one sequence term $\{nx\}$ in each of the intervals of (2.1), and will complete the proof. Writing $\alpha = \{m_1 x\} - \{m_2 x\} > 0$ (the case $\alpha < 0$ is similar), and $N_\alpha = \lfloor \frac{1}{\alpha} \rfloor$, one obtains $\{k(m_2 - m_1)x\} = \{k\left(\lfloor (m_2 - m_1)x \rfloor + 1 - \alpha\right)\} = \{k(1-\alpha)\} = 1 - \{k\alpha\} = 1 - k\alpha$, for $k = 0, \ldots, N_\alpha$ and $0 \le k\alpha \le 1$. As the distance between consecutive terms of the sequence $\{k(m_2 - m_1)x\}_{k=0}^{\infty}$ is less than $1/m$, each of the intervals of (2.1) contains at least one term. □

The following lemma describes the behavior of a sequence $\{z^n\}_{n=0}^{\infty}$ for an arbitrary value of $z \in \mathbb{C}$.

Lemma 2.2. *Let $z = re^{2\pi i x} \in \mathbb{C}$ be a complex number $(r > 0)$. The orbit of $\{z^n\}_{n=0}^{\infty}$ is*

- (i) *a regular k-gon if $r = 1$, and $x = j/k \in \mathbb{Q}$ with $gcd(j,k) = 1$;*
- (ii) *a dense subset of the unit circle for $r = 1$ and $x \in \mathbb{R} \setminus \mathbb{Q}$;*
- (iii) *an inward spiral for $r < 1$;*
- (iv) *an outward spiral for $r > 1$.*

Proof.

- (i) For $r = 1$ the terms of the sequence $\{z^n\}_{n=0}^{\infty}$ are located on the unit disk. When $x = j/k \in \mathbb{Q}$ is an irreducible fraction, z is a primitive kth root of unity. As $z^k = 1$, the sequence $\{z^n\}_{n=0}^{\infty} = \{1, z, \ldots, z^{k-1}, \ldots\}$ is periodic and describes a closed finite orbit.

  [The result is illustrated in Figure 1 for (a) $k = 5$ and (b) $k = 8$.]
- (ii) As the argument of z^n is $2\pi n x$, the principal arguments of the terms in the sequence $\{z^n\}_{n=0}^{\infty}$ form the set $\{2\pi\{nx\}\}$ which, from Lemma 2.1, is dense in the interval $[0, 2\pi]$. Thus, the orbit of $\{z^n\}_{n=0}^{\infty}$ is a dense subset of the unit circle.
- (iii) For $r < 1$ we have $\lim_{n\to\infty} \|z^n\| = \lim_{n\to\infty} \|z\|^n = \lim_{n\to\infty} r^n = 0$, therefore the sequence $\{z^n\}_{n=0}^{\infty}$ converges to the origin.

  [The terms of the sequence $\{z^n\}_{n=0}^{\infty}$ are all real for $\{x\} \in \{0, 1/2\}$. When $x = j/k \in \mathbb{Q}$ is an irreducible fraction $(k \ge 3)$ the orbit of the sequence $\{z^n\}_{n=0}^{\infty}$ forms an inward spiral whose points are aligned with the vertices of a regular polygon, as shown in

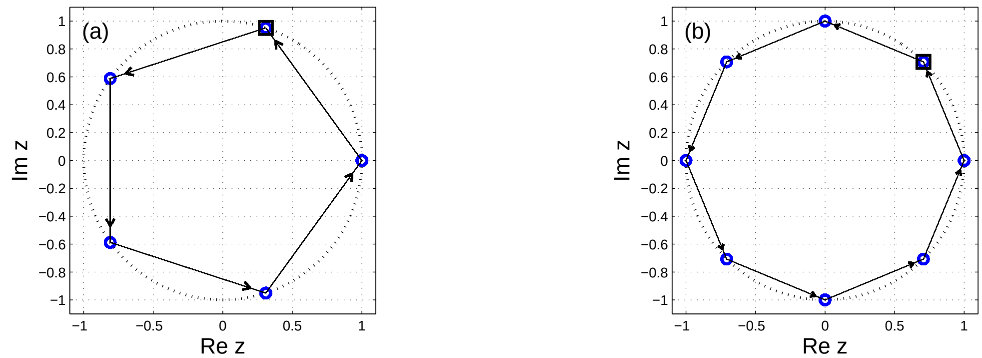

FIGURE 1. Orbit of $\{z^n\}_{n=0}^{\infty}$ obtained for $r = 1$ and (a) $x = 1/5$; (b) $x = 1/8$. Arrows indicate the direction of the orbit, and the dashed line represents the unit circle. The generator $z = r\exp(2\pi i x)$ is shown as a square.

Figure 2(a) for $x = 1/5$ and $r = 0.98$. When $x \in \mathbb{R} \setminus \mathbb{Q}$ the orbit also converges to the origin but this time the resulting points form a spiral, as in Figure 2(b) for $x = \sqrt{2}/10$ and $r = 0.98$.]

(iv) For $r > 1$ we have $\lim_{n\to\infty} \|z^n\| = \lim_{n\to\infty} \|z\|^n = \lim_{n\to\infty} r^n = \infty$, therefore the sequence $\{z^n\}_{n=0}^{\infty}$ diverges to infinity.

[The terms of the sequence $\{z^n\}_{n=0}^{\infty}$ are all real for $\{x\} \in \{0, 1/2\}$. When $x = j/k \in \mathbb{Q}$ is an irreducible fraction ($k \geq 3$) the orbit of the sequence $\{z^n\}_{n=0}^{\infty}$ forms a set of rays aligned with the vertices of a regular polygon, as depicted in Figure 3(a) for $x = 1/10$ and $r = 1.01$. When $x \in \mathbb{R}\setminus\mathbb{Q}$ the orbit also diverges to infinity but this time the points form a spiral, as illustrated in Figure 3(b) for $x = \sqrt{2}/10$ and $r = 1.01$.]

□

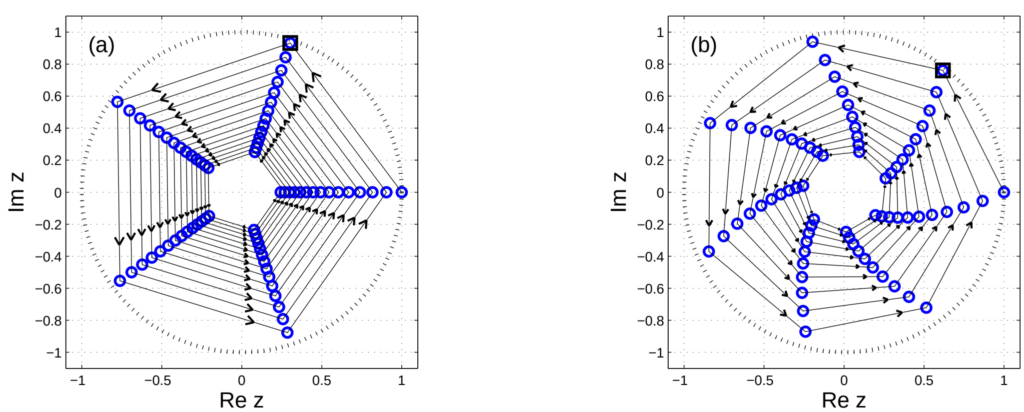

FIGURE 2. Partial orbit of $\{z^n\}_{n=0}^{\infty}$ (71 terms) obtained for $r = 0.98$ and (a) $x = 1/5$; (b) $x = \sqrt{2}/10$. Arrows indicate the direction of the orbit, and the dashed line represents the unit circle. The generator $z = r\exp(2\pi i x)$ is shown as a square.

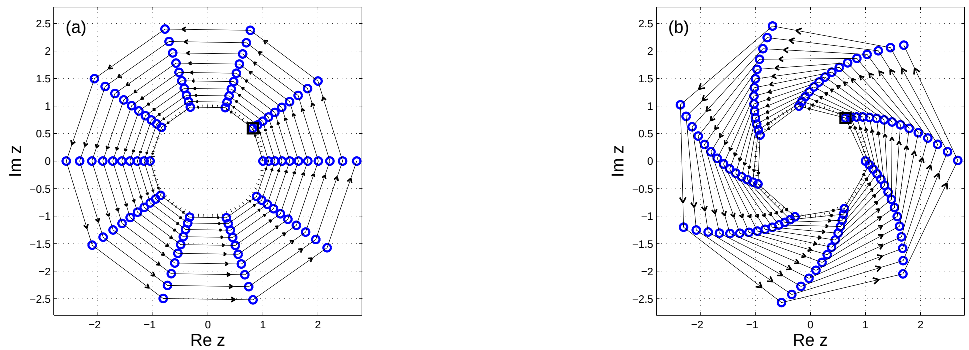

FIGURE 3. Partial orbit of $\{z^n\}_{n=0}^{\infty}$ (101 terms) obtained for $r = 1.01$ and (a) $x = 1/10$; (b) $x = \sqrt{2}/10$. Arrows indicate the direction of the orbit, and the dashed line represents the unit circle. The generator $z = r\exp(2\pi i x)$ is shown as a square.

If one of the roots of the characteristic equation (1.2) is zero then the Horadam recurrence is reduced to first order and the analysis of sequence periodicity is straightforward.

Remark 2.3. For roots z_1, z_2 of the characteristic polynomial $z^2 - pz + q$ of (1.2), $z_1 z_2 = 0$ implies $q = 0$ for which the Horadam sequence reduces to a first-order recurrence sequence. Assuming $z_2 = 0$, the general term of the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is $w_n = bz_1^{n-1}$ $(n \geq 1)$. For $z_1 \neq 0$ and $b \neq 0$ the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic if and only if the sequence $\{z_1^n\}_{n=0}^{\infty}$ is periodic, which from Lemma 2.2 is equivalent to z_1 being a kth root of unity for some $k \in \mathbb{N}$. When $b = 0$ or z_1 vanishes, the recurrence is satisfied by $w_n = 0$ $(n \geq 2)$, with the only non-zero terms being potentially w_0, w_1.

3. Main Results

For the purpose of simplification, the general Horadam sequence $\{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is written $\{w_n\}_{n=0}^{\infty}$ hereafter. In this section necessary and sufficient conditions for the periodicity of the sequence $\{w_n\}_{n=0}^{\infty}$ are established when the characteristic solutions z_1, z_2 of (1.2) are distinct or identical.

3.1. Non-Degenerate Case.

Here the conditions for periodicity are examined in the case when the characteristic solutions z_1, z_2 are distinct.

Theorem 3.1. *(Sufficient condition for periodicity.) Let $z_1 \neq z_2$ be distinct kth roots of unity $(k \geq 2)$, and let the polynomial $P(x)$ be*

$$P(x) = (x - z_1)(x - z_2), \quad x \in \mathbb{C}. \tag{3.1}$$

The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (3.1), and the arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic.

Proof. The general term of the sequence $\{w_n\}_{n=0}^{\infty}$, for distinct roots z_1, z_2 of $P(x)$, is given by (see, for example, [1, Chapter 7], [2, Chapter 1] or [7])

$$w_n = Az_1^n + Bz_2^n, \tag{3.2}$$

where A and B are constants obtained by solving the simultaneous equations

$$\begin{aligned} w_0 &= A + B = a, \\ w_1 &= Az_1 + Bz_2 = b. \end{aligned} \tag{3.3}$$

As $z_1^k = z_2^k = 1$, the sequence $\{w_n\}_{n=0}^{\infty}$ is periodic. The period is a divisor of k, and is simply $\mathrm{lcm}(\mathrm{ord}(z_1), \mathrm{ord}(z_2))$ (where $\mathrm{ord}(z)$ is the order of z). By elementary algebra,

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}, \tag{3.4}$$

so the general Horadam term can be written explicitly as

$$w_n = \frac{1}{z_2 - z_1}\left[(az_2 - b)z_1^n + (b - az_1)z_2^n\right] \tag{3.5}$$

in terms of the initial values and generators. From (3.2) one can deduce that the following degenerate (in the sense that not both generators contribute to w_n) periodic cases are possible. When $B = 0$ one obtains $b = az_1$, therefore $w_n = az_1^n$ $(n \geq 0)$, while $A = 0$ gives $w_n = az_2^n$ $(n \geq 0)$. When $A = B = 0$ we have $b = az_1 = az_2$, therefore $a = b = 0$ and $w_n = 0$ $(n \geq 0)$. □

From Lemma 2.2 it is expected that periodic sequences should employ only roots of unity, since otherwise the orbits of the sequences $\{z_1^n\}_{n=0}^{\infty}$ and $\{z_2^n\}_{n=0}^{\infty}$ have infinitely many distinct terms. A necessary condition is presented next.

Theorem 3.2. *(Necessary condition for periodicity.) Let $z_1 \neq z_2$ be the distinct roots of the characteristic polynomial (3.1). The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by z_1, z_2, and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic only if there exists $k \in \mathbb{N}$ s.t.*

$$\begin{aligned} A(z_1^k - 1)z_1 &= 0, \\ B(z_2^k - 1)z_2 &= 0, \end{aligned} \tag{3.6}$$

where A and B are given by (3.4). Explicitly, these conditions allow for the following subcases:

(i) *z_1 and z_2 are kth roots of unity (for some natural number $k \geq 2$) (non-degenerate);*
(ii) *z_1 or z_2 is a kth root of unity and the other is zero (regular polygon);*
(iii) *z_1 or z_2 is a kth root of unity and satisfies $b = az_1$ or $b = az_2$, resp. (regular polygon);*
(iv) *z_1 and z_2 are arbitrary, and $a = b = 0$ (degenerate orbit).*

Proof. Let us assume that the sequence is periodic, and let $k \in \mathbb{N}$ be the period. Under this assumption the periodicity can be expressed trivially as

$$w_n = w_{n+k}, \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

As $z_1 \neq z_2$ relations (3.2) and (3.7) give

$$w_n = Az_1^n + Bz_2^n = Az_1^{n+k} + Bz_2^{n+k} = w_{n+k}, \quad \text{for all } n \in \mathbb{N}, \tag{3.8}$$

which can further be written as

$$A(z_1^k - 1)z_1^n + B(z_2^k - 1)z_2^n = 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

The case when $0 = z_1 z_2$ (already discussed in Remark 2.3) clearly implies (3.6). Assuming that $z_1, z_2 \neq 0$ and that there exists non-zero numbers α and β such that, for all $n \in \mathbb{N}$, $\alpha z_1^n + \beta z_2^n = 0$, one can evaluate (3.9) for $n = 1, 2$ to obtain

$$-\frac{\alpha}{\beta} = \frac{z_2^2}{z_1^2} = \frac{z_2}{z_1}, \tag{3.10}$$

which is equivalent to $z_1 = z_2$ and a contradiction. The periodicity of $\{w_n\}_{n=0}^{\infty}$ gives, therefore,

$$\begin{aligned} A(z_1^k - 1) &= 0, \\ B(z_2^k - 1) &= 0, \end{aligned} \tag{3.11}$$

which implies (3.6) and completes its proof.

The only non-degenerate solution of (3.9) (in which both generators have a non-zero contribution) requires $z_1^k = z_2^k = 1$, which represents Case (i). If either A or B is zero (but not both), (3.2) shows that the sequence is determined by only one of the generators which leads to Cases (ii) and (iii). Finally, if both A and B are zero we obtain a degenerate orbit (Case (iv)). □

Theorems 3.1 and 3.2 highlight the importance of both the generators and initial values to the periodicity of the orbits of Horadam sequences; we see this again in the degenerate roots instance.

3.2. **Degenerate Case.** Here the conditions for periodicity are examined in the case when the characteristic solutions z_1, z_2 are equal.

Theorem 3.3. *(Sufficient condition for periodicity.) Let z be a kth root of unity ($k \geq 2$), and let the polynomial $P(x)$ be*

$$P(x) = (x - z)^2, \quad x \in \mathbb{C}. \tag{3.12}$$

The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (3.12), and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic when $b = az$, being otherwise divergent.

Proof. When the characteristic roots are equal ($z_1 = z_2 = z$, say) the general term of the associated Horadam sequence is given by

$$w_n = Az^n + Bnz^n, \tag{3.13}$$

where A and B are found from the equations

$$\begin{aligned} w_0 &= A = a, \\ w_1 &= (A + B)z = b; \end{aligned} \tag{3.14}$$

explicitly, then,

$$w_n = \left[a + \left(\frac{b}{z} - a\right)n\right]z^n. \tag{3.15}$$

This closed form shows that $\{w_n\}_{n=0}^{\infty}$ diverges (and is clearly not periodic) whenever $B \neq 0$. For $B = 0$ one obtains $b = az$, and in turn a periodic orbit with general term $w_n = az^n$ ($n \geq 0$), while $A = 0$ gives $w_n = bnz^{n-1}$ ($n \geq 1$) for which the sequence $\{w_n\}_{n=0}^{\infty}$ represents a (non-periodic) divergent spiral. When $A = B = 0$ we have $a = 0 = b$ and the trivial sequence with $w_n = 0$ ($n \geq 0$). □

Proposition 3.4. *When generated by a repeated kth root of unity, the terms of the divergent subsequence $\{w_{Nk+j}\}_{N=0}^{\infty}$ are collinear for each value of $j \in \{0, \ldots, k-1\}$.*

Proof. For a fixed value of $j \in \{0, \ldots, k-1\}$, the general term of $\{w_{Nk+j}\}_{N=0}^{\infty}$ from (3.13) is

$$w_{Nk+j} = [A + B(Nk + j)]z^{Nk+j} = [A + B(Nk + j)]z^j, \quad \text{for all } N \in \mathbb{N}, \tag{3.16}$$

therefore

$$w_{Nk+j} - w_j = NkBz^j, \tag{3.17}$$

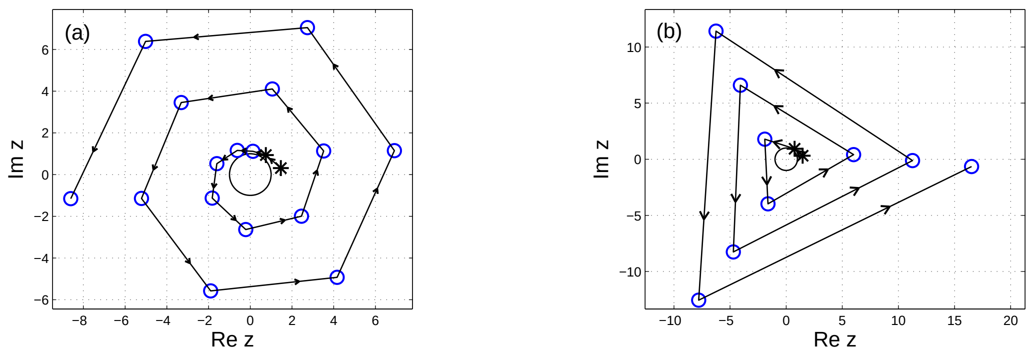

FIGURE 4. First N terms of the sequence $\{w_n\}_{n=0}^{\infty}$ for initial values $a = 1.5\exp(2\pi i/30)$, $b = 1.2\exp(2\pi i/7)$ (shown as stars) for (a) $N = 18$, $z = e^{2\pi i \frac{1}{6}}$; (b) $N = 11$, $z = e^{2\pi i \frac{2}{6}}$. Arrows indicate the direction of the sequence trajectory.

whose argument is independent of N. This proves that the terms of the (infinite) subsequence $\{w_{Nk+j}\}_{N=0}^{\infty}$ are collinear for every value of $j \in \{0, \dots, k-1\}$, as depicted in Figure 4 where sequence terms are calculated from (3.15). $\square$

The number of rays formed by aligned sequence terms for each value of $j \in \{0, \dots, k-1\}$ is equal to $k/\gcd(j, k)$, as seen in Figure 4 for (a) $k = 6, j = 1$ and (b) $k = 6, j = 2$.

Theorem 3.5. *(Necessary condition for periodicity.) The recurrence sequence $\{w_n\}_{n=0}^{\infty}$ generated by the characteristic polynomial (3.12), and arbitrary initial values $w_0 = a$, $w_1 = b$, is periodic only if one of the following is true:*

$$\begin{aligned} z &= 0, \\ z^k - 1 &= 0, \quad B = 0, \\ z^k - 1 &\neq 0, \quad A = B = 0. \end{aligned} \tag{3.18}$$

Explicitly, these conditions give the subcases

(i) $z = 0$ *(degenerate orbit);*
(ii) z *is a* k*th root of unity (for some natural number* $k \geq 2$*) and* $b = az$ *(regular polygon);*
(iii) z *is arbitrary and* $a = b = 0$ *(degenerate orbit).*

Proof. For $z_1 = z_2 = z$, the general term (3.13) and periodicity property (3.7) together give

$$w_n = Az^n + Bnz^n = Az^{n+k} + B(n+k)z^{n+k} = w_{n+k}, \qquad \text{for all } n \in \mathbb{N}. \tag{3.19}$$

This can be written as

$$\left[A(z^k - 1) + Bkz^k + Bn(z^k - 1)\right] z^n = 0, \qquad \text{for all } n \in \mathbb{N}. \tag{3.20}$$

The case $z = 0$ (already discussed in Remark 2.3) clearly implies (3.18). When $z \neq 0$ one has

$$A(z^k - 1) + Bkz^k + Bn(z^k - 1) = 0, \qquad \text{for all } n \in \mathbb{N}. \tag{3.21}$$

As this is true for every value of n, the l.h.s. linear polynomial in n is null, hence

$$\begin{aligned} A(z^k - 1) + Bkz^k &= 0, \\ B(z^k - 1) &= 0. \end{aligned} \tag{3.22}$$

It follows readily that if $z^k - 1 = 0$ we have $B = 0$, while $z^k - 1 \neq 0$ implies $A = B = 0$. The only non-degenerate case occurs, therefore, when $b = az$, with z a kth root of unity ($k \geq 2$). This ends the proof. □

4. Further Results

To finish, we use elementary properties of complex numbers to prescribe inner and outer boundaries of those regions containing the orbits of periodic Horadam sequences.

Theorem 4.1. *When the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is periodic, the orbit is subject to the following geometric boundaries:*

(i) *For $z_1 \neq z_2$ the orbit is located inside the annulus*

$$\{z \in \mathbb{C} : |\,|A| - |B|\,| \leq |w_n| \leq |A| + |B|\}, \quad \textit{for all } n \in \mathbb{N}, \tag{4.1}$$

where the constants A and B are given by (3.4);

(ii) *For $z_1 = z_2 = z$ the orbit is either a subset (regular k-gon) of the circle*

$$S(0, |a|) = \{z \in \mathbb{C} : |z| = |a|\} \tag{4.2}$$

for $a \neq 0$, or else the zero set $\{0\}$ for $a = 0$.

Proof. Any two complex numbers u and v satisfy the well-known triangle inequalities

$$|\,|u| - |v|\,| \leq |u + v| \leq |u| + |v|. \tag{4.3}$$

(i) For $z_1 \neq z_2$ the general term of the sequence (3.2), combined with (4.3), gives

$$|\,|Az_1^n| - |Bz_2^n|\,| \leq |w_n| = |Az_1^n + Bz_2^n| \leq |Az_1^n| + |Bz_2^n|, \tag{4.4}$$

which as $|z_1| = |z_2| = 1$ (from the Theorem 3.2 periodicity condition) is equivalent to

$$|\,|A| - |B|\,| \leq |w_n| = |Az_1^n + Bz_2^n| \leq |A| + |B|. \tag{4.5}$$

[We illustrate this result in Figure 5 for sequences obtained when z_1 and z_2 are (a) 5th roots and (b) 6th roots of unity. One should note that in the proof we have only used the fact that z_1 and z_2 lie on the unit circle.]

(ii) When $z_1 = z_2 = z$ then, from Theorem 3.5, the sequence can be periodic only when $B = 0$ and z is a kth primitive root, in which case $w_n = az^n$ ($n \geq 0$) and the orbit is a regular k-gon with $|w_n| = |a|$; when a is also zero the sequence terms vanish.

□

5. Summary

It is almost half a century since, in one of his initial publications [5], Horadam himself made a passing remark about two periodic p, q sequence instances $\{w_n(a, b; \pm 1, 1)\}_{n=0}^{\infty}$. Since that time—as evident from the survey article [8]—the notion of periodicity in Horadam sequences has, somewhat surprisingly, not been given any proper scrutiny until now.

In developing the results given here, it has become clear that there are a variety of interesting facets to the self-repeating behavior of such sequences which are of mathematical interest *per se* and whose underpinning theoretical basis yields potential applications in computing. Such

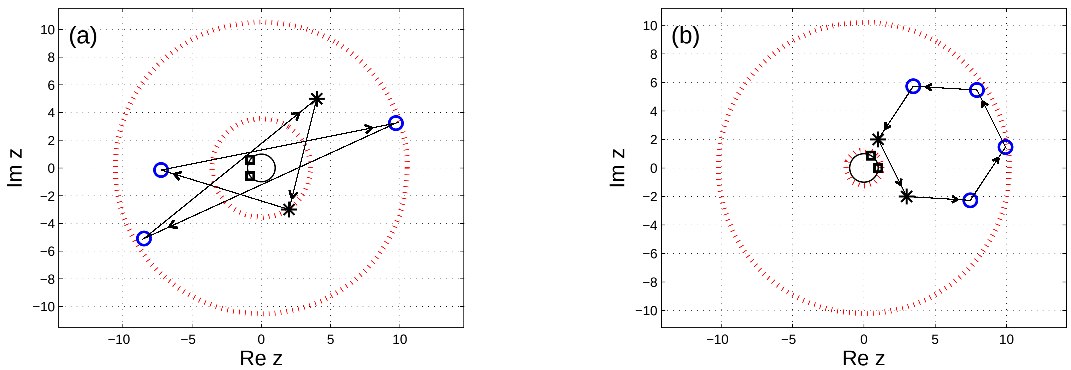

FIGURE 5. Orbit of a periodic Horadam sequence $\{w_n\}_{n=0}^{\infty}$ computed for (a) $z_1 = e^{2\pi i \frac{2}{5}}$, $z_2 = e^{2\pi i \frac{3}{5}}$, $a = 4 + 5i$, $b = 2 - 3i$; (b) $z_1 = e^{2\pi i \frac{1}{6}}$, $z_2 = e^{2\pi i \frac{5}{6}}$, $a = 1 + 2i$, $b = 3 - 2i$. Also plotted are the initial values a, b (stars), the generators z_1, z_2 (squares), the unit circle $S(0,1)$ (solid line) and boundaries of the annulus $U(0, ||A| - |B||, |A| + |B|)$ (dashed lines).

work, however, lies beyond the remit of this particular paper which serves merely to introduce the reader to the concept of Horadam cyclicity and its salient governing conditions. Regarding further analytical work planned, this includes examining in detail geometrical aspects of periodic complex orbits, and looking separately at the issue of periodicity through a matrix-based approach which appeals to the theory of eigenvectors and eigenvalues. There also remains, of course, the possibility that results on cyclicity might be formulated for a generalized Horadam sequence which satisfies a linear recurrence of arbitrary order.

Acknowledgement

The constructive comments of the referee—to whom we extend our gratitude—have improved the readability of the paper.

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MSC2010: 11K31, 11B39

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