ON INVERSE RELATIONS FOR GENERAL LUCAS SEQUENCES OF POLYNOMIALS

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Abstract. We obtain inverse relations for the definitional forms of general Lucas sequences of polynomials in a single variable.

1. Introduction

This communication was motivated by the following interesting problem that was proposed by P. Bruckman [2] and was recently solved by Á. Plaza and S. Falcón [5]. If the Fibonacci polynomials, $F_n(x)$, are defined by

$$F_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k}, n = 0, 1, 2, \ldots \tag{1.1}$$

then establish the following "inverse" relation of (1.1):

$$x^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{n+1-2k}(x), n = 0, 1, 2, \ldots. \tag{1.2}$$

The term "inverse relation" of equation (1.1) here means the representation of the powers of x as linear combinations of elements of the Lucas sequence of the Fibonacci polynomials. In the present note we extend (and modify, slightly) the above result to more general Lucas sequences of polynomials.

2. Definitions and Lemmas

Let $x \in \mathbb{R}$ and let $P, Q \in \mathbb{Z}$ be nonzero throughout this paper. For Fibonacci polynomials, $P = 1$ and $Q = -1$; by analogy to equation (1.1), we *define* the sequence

$$U_{n+1}(x; P, Q) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (Px)^{n-2k} (-Q)^k, n = 0, 1, 2 \ldots, \tag{2.1}$$

where $[n/2]$ denotes the largest integer not greater than $n/2$, and we define 0^0 to be 1. Henceforth, we write, for simplicity, $U_n(x)$ in place of $U_n(x; P, Q)$. For each $n \in \mathbb{N}$, $U_n(x)$ is a polynomial in x of degree $n - 1$ (if $x \neq 0$): $U_1(x) = 1$, $U_2(x) = Px$, $U_3(x) = (Px)^2 - Q$, and so on.

It is straightforward to show that the sequence $\{U_n(x)\}_{n=1}^{\infty}$ in equation (2.1) can just as well be *defined* by the linear recurrence relation of order 2:

$$U_n(x) = \begin{cases} 1 & n = 1, \\ Px & n = 2, \\ (Px)U_{n-1}(x) - QU_{n-2}(x) & n \geq 3. \end{cases} \tag{2.2}$$

Since the linear recurrence relation in equation (2.2) is of the same form as that which characterizes Lucas sequences of constants, we *define* the sequence described by equation (2.2) (for a particular choice of parameters P and Q) as the Lucas sequence of polynomials in x associated to the pair (P, Q) [6, pp. 36-37]. The special cases of $(P, Q) = (1, -1), (2, -1)$ enjoy wide interest [1, 3, 7].

3. The Main Theorem

The Binet formulation is a well-known way of defining the Fibonacci numbers [4, pp. 147-148]. Generalizing it to the present situation, we let $x \in \mathbb{R}$ be such that $(Px)^2 - 4Q = D > 0$, and then let the two roots of the quadratic $y^2 - (Px)y + Q = 0$ be denoted by $\alpha(x)$ and $\beta(x)$, where $\alpha(x) > \beta(x)$ by convention and the inequality is strict. We obtain the simple relations

$$\begin{array}{ll} \alpha(x) = (Px + \sqrt{D})/2 & \alpha(x) + \beta(x) = Px \\ \beta(x) = (Px - \sqrt{D})/2 & \alpha(x) \cdot \beta(x) = \ Q\ . \end{array}$$

To remove clutter, we write $\alpha \cong \alpha(x)$ and $\beta \cong \beta(x)$, remembering that α, β here (and, henceforth, in the paper) are *functions* of x. The first two members of the sequence

$$\left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}_{n=1}^{\infty} \tag{3.1}$$

are, in fact, $U_1(x)$ and $U_2(x)$. It is straightforward to show that for any $n \geq 3$, one has the linear recurrence relation of order 2

$$(Px)\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) - Q\left(\frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta}\right) = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{3.2}$$

It follows that for every $n \geq 1$ one has $U_n(x) = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$.

It is clear, inductively, from equation (2.1) that even- (odd-) indexed $U_n(x)$'s contain only odd (even) powers of x, and that an analog of equation (1.2) might have the form

$$(Px)^n = \sum_{k=0}^{[n/2]} C_k U_{n+1-2k}(x), \tag{3.3}$$

where the C_k's remain to be determined. In the Binet formulation, equation (3.3) is equivalent to

$$(\alpha - \beta)(\alpha + \beta)^n = \sum_{k=0}^{[n/2]} C_k [\alpha^{n+1-2k} - \beta^{n+1-2k}]. \tag{3.4}$$

Theorem 3.1. *For any nonnegative n, an inverse relation of equation (2.1) is*

$$(Px)^n = \sum_{k=0}^{[n/2]} \left[\frac{n+1-2k}{n+1}\right] \binom{n+1}{k} Q^k U_{n+1-2k}(x). \tag{3.5}$$

Proof. The theorem is trivial for $n = 0, 1$, since $(Px)^0 = 1 = U_1(x)$ and $(Px)^1 = U_2(x)$ from equation (2.2). Now suppose that $n > 1$ is odd; then one has

$$\begin{aligned}
(\alpha-\beta)(\alpha+\beta)^n &= \sum_{k=0}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^k - \sum_{K=0}^{n}\binom{n}{K}\alpha^{n-K}\beta^{K+1} \\
&= \alpha^{n+1}-\beta^{n+1} + \sum_{k=1}^{n}\binom{n}{k}\alpha^{n+1-k}\beta^k - \sum_{k=1}^{n}\binom{n}{k-1}\alpha^{n+1-k}\beta^k \quad (k=K+1) \\
&= \alpha^{n+1}-\beta^{n+1} + \sum_{k=1}^{(n-1)/2}\left[\binom{n}{k}-\binom{n}{k-1}\right](\alpha\beta)^k\alpha^{n+1-2k} \\
&\quad + \sum_{K=(n+1)/2}^{n}\left[\binom{n}{K}-\binom{n}{K-1}\right]\alpha^{n+1-K}\beta^K \\
&= \alpha^{n+1}-\beta^{n+1} + \sum_{k=1}^{(n-1)/2}\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}(\alpha\beta)^k\alpha^{n+1-2k} \\
&\quad + \sum_{k=1}^{(n-1)/2} -\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}(\alpha\beta)^k\beta^{n+1-2k}, \qquad (k=n+1-K)
\end{aligned}$$

where the last summation terminates at $k = (n-1)/2$ because the putative last term at $k = (n+1)/2$ would be 0. We obtain

$$(\alpha-\beta)(\alpha+\beta)^n = \alpha^{n+1}-\beta^{n+1} + \sum_{k=1}^{(n-1)/2}\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}(\alpha\beta)^k[\alpha^{n+1-2k}-\beta^{n+1-2k}],$$

and division by $\alpha-\beta$ yields, finally,

$$(Px)^n = \sum_{k=0}^{[n/2]}\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}Q^kU_{n+1-2k}(x).$$

On the other hand, if $n > 0$ is even, then

$$\begin{aligned}(\alpha-\beta)(\alpha+\beta)^n &= \alpha^{n+1}-\beta^{n+1}+\sum_{k=1}^{n/2}\left[\binom{n}{k}-\binom{n}{k-1}\right]\alpha^{n+1-k}\beta^k \\ &\quad -\sum_{K=(n+2)/2}^{n}\left[\binom{n}{K-1}-\binom{n}{K}\right]\alpha^{n+1-K}\beta^K \\ &= \alpha^{n+1}-\beta^{n+1}+\sum_{k=1}^{n/2}\left[\binom{n}{k}-\binom{n}{k-1}\right](\alpha^{n+1-k}\beta^k-\alpha^k\beta^{n+1-k}) \\ &\qquad\qquad (k=n+1-K) \\ &= \alpha^{n+1}-\beta^{n+1}+\sum_{k=1}^{n/2}\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}(\alpha\beta)^k[\alpha^{n+1-2k}-\beta^{n+1-2k}] \\ &= \sum_{k=0}^{[n/2]}\left[\frac{n+1-2k}{n+1}\right]\binom{n+1}{k}Q^k[\alpha^{n+1-2k}-\beta^{n+1-2k}],\end{aligned}$$

and division by $(\alpha-\beta)$ again gives equation (3.5). □

For example, with $n = 5$, one obtains from equation (3.5)

$$(Px)^5 = \sum_{k=0}^{2}\left[\frac{6-2k}{6}\right]\binom{6}{k}Q^kU_{6-2k}(x) = U_6(x)+4QU_4(x)+5Q^2U_2(x).$$

It is observed that equation (3.5) is a greedier inverse relation than is the analogous one in equation (1.2). For example, equation (3.5) uses only 6 terms when $n = 10$, but equation (1.2) requires 11 terms.

It is also observed, as expected and desired, that if equation (2.1) is substituted into equation (3.5), one obtains the tautology $(Px)^n = (Px)^n$. Similarly, substitution of equation (3.5) into equation (2.1) yields the tautology $U_{n+1}(x) = U_{n+1}(x)$. These observations justify the use of the term "inverse relation." The formal proofs are left to the reader.

4. The Companion Lucas Sequence of Polynomials

In the spirit of Binet, we *define* the companion Lucas sequence of polynomials in x associated to the pair (P, Q) to be the sequence

$$\{V_n(x)\}_{n=0}^{\infty}, \quad V_n(x) = \alpha^n + \beta^n, \tag{4.1}$$

where the symbols α, β have the same meaning as in equation (3.1). That equation (4.1) really is a sequence of polynomials in x follows from the facts that $V_0(x) = 2$, $V_1(x) = Px$, and equation (4.1) obeys the same linear recurrence relation of order 2 as does equation (3.1). Mathematical induction shows that odd-(even-) indexed $V_n(x)$'s contain only terms of odd (even) degree.

It is of interest to work out the inverse relations of the polynomial representations of the $V_n(x)$'s, analogous to Theorem 3.1. The easiest way first to see what the $V_n(x)$'s look like is to connect them to the $U_n(x)$'s; we require two lemmas.

Lemma 4.1. *For any $n \in \mathbb{N}$, $(Px)U_n(x) - 2QU_{n-1}(x) = V_n(x)$.*

Proof. The lemma holds if and only if

$$(\alpha+\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta}-2\alpha\beta\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=\alpha^n+\beta^n,$$

and this follows almost immediately by expansion. □

Lemma 4.2. *For* $0\le 2k\le n-1$,

$$\binom{n-1-k}{k}+2\binom{n-1-k}{k-1}=\left[\frac{n}{n-k}\right]\binom{n-k}{k}.$$

Proof.

$$\binom{n-1-k}{k}+2\binom{n-1-k}{k-1}=\frac{(n-k)!(n-2k)}{(n-k)k!(n-2k)!}+\frac{2(n-k)!k}{(n-k)k!(n-2k)!}$$

□

Theorem 4.1. *Let* x,P,Q *be as stated in Section 2 and let* $(Px)^2-4Q>0$. *Then we have*

$$V_n(x)=\begin{cases}2, & n=0,\\ \displaystyle\sum_{k=0}^{[n/2]}\left[\frac{n}{n-k}\right]\binom{n-k}{k}(Px)^{n-2k}(-Q)^k, & n>0.\end{cases}\tag{4.2}$$

Proof. The theorem is trivial for $n=0$. If n is odd, then equation (2.1) and Lemmas 4.1 and 4.2 give

$$\begin{aligned}V_n(x)&=(Px)\sum_{k=0}^{[(n-1)/2]}\binom{n-1-k}{k}(Px)^{n-1-2k}(-Q)^k\\&\quad+2\sum_{K=0}^{[(n-2)/2]}\binom{n-2-K}{K}(Px)^{n-2-2K}(-Q)^{K+1}\\&=(Px)^n+\sum_{k=1}^{(n-1)/2}\binom{n-1-k}{k}(Px)^{n-2k}(-Q)^k\\&\quad+\sum_{k=1}^{(n-1)/2}2\binom{n-1-k}{k-1}(Px)^{n-2k}(-Q)^k \qquad (k=K+1)\\&=(Px)^n+\sum_{k=1}^{(n-1)/2}\left[\frac{n}{n-k}\right]\binom{n-k}{k}(Px)^{n-2k}(-Q)^k\\&=\sum_{k=0}^{[n/2]}\left[\frac{n}{n-k}\right]\binom{n-k}{k}(Px)^{n-2k}(-Q)^k.\end{aligned}$$

If $n > 0$ is even, then one obtains

$$\begin{aligned} V_n(x) &= \left[(Px)^n + \sum_{k=1}^{(n-2)/2} \binom{n-1-k}{k} (Px)^{n-2k}(-Q)^k\right] \\ &\quad + \left[\sum_{K=0}^{(n-4)/2} 2\binom{n-2-K}{K}(Px)^{n-2-2K}(-Q)^{K+1} + 2(-Q)^{n/2}\right] \\ &= (Px)^n + \sum_{k=1}^{(n-2)/2} \left[\frac{n}{n-k}\right] \binom{n-k}{k} (Px)^{n-2k}(-Q)^k + 2(-Q)^{n/2} \\ &= \sum_{k=0}^{[n/2]} \left[\frac{n}{n-k}\right] \binom{n-k}{k} (Px)^{n-2k}(-Q)^k. \end{aligned}$$

□

For comparison, below we give a short listing of the polynomials $U_n(x)$ and $V_n(x)$; several patterns in the second and third columns seem apparent.

n	$U_n(x)$	$V_n(x)$
0	0	2
1	1	Px
2	Px	$(Px)^2 - 2Q$
3	$(Px)^2 - Q$	$(Px)^3 - 3(Px)Q$
4	$(Px)^3 - 2(Px)Q$	$(Px)^4 - 4(Px)^2Q + 2Q^2$
5	$(Px)^4 - 3(Px)^2Q + Q^2$	$(Px)^5 - 5(Px)^3Q + 5(Px)Q^2$
6	$(Px)^5 - 4(Px)^3Q + 3(Px)Q^2$	$(Px)^6 - 6(Px)^4Q + 9(Px)^2Q^2 - 2Q^3$
7	$(Px)^6 - 5(Px)^4Q + 6(Px)^2Q^2 - Q^3$	$(Px)^7 - 7(Px)^5Q + 14(Px)^3Q^2 - 7(Px)Q^3$

An analog of relation (3.3) might assume the form

$$(Px)^n = \sum_{k=0}^{[n/2]} C_k V_{n-2k}(x), \tag{4.3}$$

where the C_k's are to be determined. In the Binet formulation, this would be

$$(\alpha+\beta)^n = \sum_{k=0}^{[n/2]} C_k(\alpha^{n-2k} + \beta^{n-2k}). \tag{4.4}$$

Some care is needed here, however, because the parity of n may make a difference.

Theorem 4.2. *Let x, P, Q be as stated in Section 2 and let $(Px)^2 - 4Q > 0$. Then we have*

$$(Px)^n = \sum_{k=0}^{[n/2]} \binom{n}{k} Q^k V_{n-2k}(x) - \begin{cases} 0 & n = odd \\ \frac{1}{2}\binom{n}{n/2} Q^{n/2} V_0(x) & n = even. \end{cases} \tag{4.5}$$

Proof. For odd $n \geq 1$, we have from the identity $\binom{n}{k} = \binom{n}{n-k}$,

$$(\alpha+\beta)^n = \alpha^n + \beta^n + \sum_{k=1}^{(n-1)/2} \binom{n}{k} (\alpha\beta)^k (\alpha^{n-2k} + \beta^{n-2k}),$$

that is,

$$(Px)^n = V_n(x) + \sum_{k=1}^{(n-1)/2} \binom{n}{k} Q^k V_{n-2k}(x) = \sum_{k=0}^{(n-1)/2} \binom{n}{k} Q^k V_{n-2k}(x). \tag{4.6}$$

The case $n = 0$ is trivial: $(Px)^0 = 1 = \frac{1}{2} V_0(x)$. Now suppose that $n > 0$ is even. Binomial expansion again gives

$$(\alpha+\beta)^n = \alpha^n + \beta^n + \sum_{k=1}^{(n-2)/2} \binom{n}{k} (\alpha\beta)^k (\alpha^{n-2k} + \beta^{n-2k}) + \frac{1}{2}\binom{n}{n/2} (\alpha\beta)^{n/2} (\alpha^0 + \beta^0),$$

and we observe that if n were 0, the last term would be $\frac{1}{2}V_0(x)$. Hence, for all even $n \geq 0$,

$$(Px)^n = \sum_{k=0}^{n/2} \binom{n}{k} Q^k V_{n-2k}(x) - \frac{1}{2}\binom{n}{n/2} Q^{n/2} V_0(x). \tag{4.7}$$

Finally, equations (4.6) and (4.7) may be joined to give equation (4.5). □

Again, for comparison, we give a short listing of inverse relations expressed as linear combinations of either the $U_n(x)$'s (column 2) or the $V_n(x)$'s (column 3).

$(Px)^n$

n	$U_n(x)$'s	$V_n(x)$'s
0	$U_1(x)$	$\frac{1}{2}V_0(x)$
1	$U_2(x)$	$V_1(x)$
2	$U_3(x) + QU_1(x)$	$V_2(x) + QV_0(x)$
3	$U_4(x) + 2QU_2(x)$	$V_3(x) + 3QV_1(x)$
4	$U_5(x) + 3QU_3(x) + 2Q^2U_1(x)$	$V_4(x) + 4QV_2(x) + 3Q^2V_0(x)$
5	$U_6(x) + 4QU_4(x) + 5Q^2U_2(x)$	$V_5(x) + 5QV_3(x) + 10Q^2V_1(x)$
6	$U_7(x) + 5QU_5(x) + 9Q^2U_3(x) + 5Q^3U_1(x)$	$V_6(x) + 6QV_4(x) + 15Q^2V_2(x) + 10Q^3V_0(x)$
7	$U_8(x) + 6QU_6(x) + 14Q^2U_4(x) + 14Q^3U_2(x)$	$V_7(x) + 7QV_5(x) + 21Q^2V_3(x) + 35Q^3V_1(x)$

It is apparent that when $Q = 1$, the sum of the coefficients in column 3, corresponding to a given n, is 2^{n-1}. Less obvious, but also convincing, is the assertion that the analogous sum in column 2, corresponding to a given n, is $\binom{n}{(n+1)/2}$ if n is odd and $\binom{n}{n/2}$ if n is even. Readers may spot other trends.

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References

[1] M. Bicknell, *A primer for the Fibonacci numbers. VII: Introduction to Fibonacci polynomials and their divisibility properties*, The Fibonacci Quarterly, **8.4** (1970), 407–420.
[2] P. S. Bruckman, *Problem B-1075*, The Fibonacci Quarterly, **48.3** (2010), 278.
[3] W. Chu and N. N. Li, *Power sums of Pell and Pell-Lucas polynomials*, The Fibonacci Quarterly, **49.2** (2011), 139–150.
[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Oxford University Press, Oxford, 1979.
[5] Á. Plaza and S. Falcón, *Solution of Problem B-1075*, The Fibonacci Quarterly, **49.4** (2011), 369-370.
[6] P. Ribenboim, *The Little Book of Big Primes,* Springer-Verlag, New York, 1991.
[7] Y. Yuan and W. Zhang, *Some identities involving the Fibonacci polynomials*, The Fibonacci Quarterly, **40.4** (2002), 314–318.

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