ON THE MODES OF THE POISSON DISTRIBUTION OF ORDER K

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Abstract. Sharp upper and lower bounds are established for the modes of the Poisson distribution of order k. The lower bound established in this paper is better than the previously established lower bound. In addition, for $k = 2, 3, 4, 5$, a recent conjecture is presently proved solving partially an open problem since 1983.

1. Introduction and Summary

For any given positive integer k, denote by N_k the number of independent trials with constant success probability p until the occurrence of the kth consecutive success, and set $q = 1 - p$. For $n \geq k$, Philippou and Muwafi [13] derived the probability $P(N_k = n)$ in terms of multinomial coefficients and noted that $P(N_k = n \,|\, p = 1/2) = f^{(k)}_{n-k+1}/2^n$ where $f^{(k)}_n$ is the nth Fibonacci number of order k, [3, 15, 16]. Philippou, et al. [12] showed that $\sum_{n=k}^{\infty} P(N_k = n) = 1$ and named the distribution of N_k *the geometric distribution of order k with parameter p*, since for $k = 1$ it reduces to the geometric distribution with parameter p. Assuming that $N_{k,1}, \dots, N_{k,r}$ are independent random variables distributed as geometric of order k with parameter p, and setting $Y_{k,r} = \sum_{j=1}^{r} N_{k,j}$, the latter authors showed that

$$P(Y_{k,r} = y) = p^y \sum \binom{y_1 + \cdots + y_k + r - 1}{y_1, \dots, y_k, r - 1} \left(\frac{q}{p}\right)^{y_1+\cdots+y_k} \qquad y = kr, kr + 1, \dots,$$

where the summation is taken over all k-tuples of nonnegative integers $y_1, y_2, \dots, y_k$ such that $y_1 + 2y_2 + \cdots + ky_k = y - kr$. They named the distribution of $Y_{k,r}$ *the negative binomial distribution of order k with parameters r and p*, since for $k = 1$ it reduces to the negative binomial distribution with the same parameters. Furthermore they showed that, if $rq \to \lambda\,(\lambda > 0)$ as $r \to \infty$ and $q \to 0$, then

$$\lim_{r\to\infty} P(Y_{k,r} - kr = x) = \sum_{x_1,\dots,x_k} e^{-k\lambda} \frac{\lambda^{x_1+x_2+\cdots+x_k}}{x_1! \cdots x_k!} = f_k(x; \lambda), \quad x = 0, 1, 2, \dots, \tag{1.1}$$

where the summation is taken over all k-tuples of nonnegative integers $x_1, x_2, \dots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$. They named the distribution with probability mass function $f_k(x; \lambda)$ *the Poisson distribution of order k with parameter λ*, since for $k = 1$ it reduces to the Poisson distribution with parameter λ, [1, 9, 2].

Denote by $m_{k,\lambda}$ the mode(s) of $f_k(x; \lambda)$, i.e. the value(s) of x for which $f_k(x; \lambda)$ attains its maximum. It is well-known that $m_{1,\lambda} = \lambda$ or $\lambda - 1$ if $\lambda \in \mathbb{N}$, and $m_{1,\lambda} = \lfloor \lambda \rfloor$ if $\lambda \notin \mathbb{N}$. Philippou [7] derived some properties of $f_k(x; \lambda)$ and posed the problem of finding its mode(s) for $k \geq 2$. See also [8] and [11].

Hirano, et al. [5] presents several graphs of $f_k(x; \lambda)$ for $\lambda \in (0, 1)$ and $2 \leq k \leq 8$, and Luo [6] derived the following inequality

$$m_{k,\lambda} \geq k\lambda \sqrt[k]{k!} - \frac{k(k+1)}{2}, \quad k \geq 1 \ (\lambda > 0), \tag{1.2}$$

which is sharp in the sense that $m_{1,\lambda} = \lambda - 1$ for $\lambda \in \mathbb{N}$. Recently, Philippou and Saghafi [14] conjectured that, for $k \geq 2$ and $\lambda \in \mathbb{N}$,

$$m_{k,\lambda} = \lambda k(k+1)/2 - \lfloor k/2 \rfloor, \tag{1.3}$$

where $\lfloor u \rfloor$ denotes the greatest integer not exceeding $u \in \mathbb{R}$.

In this paper, we employ the probability generating function of the Poisson distribution of order k to improve the bound of Luo [6] and to also give an upper bound (see Theorem 2.1). We then use Theorem 2.1 to prove the conjecture of Philippou and Saghafi [14] when $k = 2, 3, 4, 5$, partially answering the open problem of Philippou [7, 8, 11].

2. Main Results

In the present section, we state and prove the following two theorems.

Theorem 2.1. *For any integer $k \geq 1$ and real $\lambda > 0$, the mode of the Poisson distribution of order k satisfies the inequalities*

$$\lfloor \lambda k(k+1)/2 \rfloor - \frac{k(k+1)}{2} + 1 - \delta_{k,1} \leq m_{k,\lambda} \leq \lfloor \lambda k(k+1)/2 \rfloor,$$

where $\delta_{k,1}$ is the Kronecker delta.

Theorem 2.2. *For $\lambda \in \mathbb{N}$ and $2 \leq k \leq 5$, the Poisson distribution of order k has a unique mode $m_{k,\lambda} = \lambda k(k+1)/2 - \lfloor k/2 \rfloor$.*

For the proofs of the theorems we employ the probability generating function of the Poisson distribution of order k and some recurrences derived from it. We observe first that the left-hand side inequality in Theorem 2.1 is sharp since, for $\lambda \in \mathbb{N}, m_{1,\lambda} = \lambda - 1$, the value of the lower bound for $k = 1$. The right-hand side inequality is also sharp in the sense that there exist values of k and λ for which $m_{k,\lambda} = \lfloor \lambda k(k+1)/2 \rfloor$. We also note that our lower bound is better than that of Luo [6] for $k \geq 2$.

Proof of Theorem 2.1. For notational simplicity, we presently set $P_x = f_k(x; \lambda)$, omitting the dependence on k and λ, and $\Delta_x = P_x - P_{x-1}$, $x = 0, 1, \ldots$. It is easily seen [4, 7, 10] that the probability generating function of P_x is

$$g(s) = \sum_{x=0}^{\infty} s^x P_x = e^{\lambda(-k+s+s^2+\cdots+s^k)}, \tag{2.1}$$

which implies that

$$g'(s) = \lambda(1 + 2s + \cdots + ks^{k-1})g(s). \tag{2.2}$$

For $x \geq 1$, we differentiate $(x-1)$ times $g'(s)$ and employ the fact that $P_x = \left(\frac{1}{x!}\right) \frac{\partial^x g(s)}{\partial s^x}$ at $s = 0$ to get the recurrence

$$xP_x = \sum_{j=1}^{k} j\lambda P_{x-j}, \quad x \geq 1. \tag{2.3}$$

We note that (2.3) is trivially true for $x = 0$. By definition $P_x \le P_{m_{k,\lambda}}$ for every $x \ge 0$, and therefore

$$xP_x = \sum_{j=1}^{k} j\lambda P_{x-j} \le \sum_{j=1}^{k} j\lambda P_{m_{k,\lambda}} = \lambda P_{m_{k,\lambda}} k(k+1)/2.$$

Upon setting $x = m_{k,\lambda}$ we get $m_{k,\lambda} \le \lambda k(k+1)/2$. Therefore, $m_{k,\lambda} \le \lfloor \lambda k(k+1)/2 \rfloor$ since $m_{k,\lambda}$ is a nonnegative integer.

As for the left-hand side inequality we note that it is trivially true for $k = 1$ and $\lambda > 0$, since $m_{1,\lambda} = \lambda$ or $\lambda - 1$ if $\lambda \in \mathbb{N}$, and $m_{1,\lambda} = \lfloor \lambda \rfloor$ if $\lambda \notin \mathbb{N}$. Therefore we assume that $k \ge 2$. For $0 < \lambda < 1$, the inequality is true since $\lfloor \lambda k(k+1)/2 \rfloor - \frac{k(k+1)}{2} + 1 \le 0$. For $\lambda = 1$ it is also true since $e^{-k} = P_0 = P_1 < P_2 = 3e^{-k}/2$. Let then $\lambda > 1$. We will show that P_x increases, or, equivalently, Δ_x is positive, for $0 \le x \le \lfloor \lambda k(k+1)/2 \rfloor - \frac{k(k+1)}{2} + 1$.

From the definition of Δ_x and equation (2.1), we obtain

$$h(s) = \sum_{x=0}^{\infty} s^x \Delta_x = (1-s)g(s). \tag{2.4}$$

Differentiating $h(s)$ twice we get

$$h''(s) = \lambda \left(\lambda \sum_{j=1}^{k-1} \frac{j(j+1)}{2} s^{j-1} + \left(\frac{k(k+1)}{2} \right) (\lambda - 2) s^{k-1} + s^k f(s) \right) g(s), \tag{2.5}$$

where $f(s) = \sum_{j=0}^{k-1} a_j s^j$ is a $(k-1)$th degree polynomial. Next, differentiating x times $h''(s)$ and then setting $s = 0$, we get

$$\tfrac{(x+1)(x+2)}{\lambda} \Delta_{x+2} = \sum_{j=1}^{k-1} \tfrac{j(j+1)}{2} \lambda P_{x+1-j} + \tfrac{k(k+1)}{2} (\lambda - 2) P_{x-k+1} + \sum_{j=0}^{k-1} a_j P_{x-k-j}.$$

Finally, eliminating successively $P_{x-2k+1}, P_{x-2k}, \ldots, P_{x-k}$, by means of equation (2.3) we arrive at

$$\tfrac{(x+1)(x+2)}{\lambda} \Delta_{x+2} = \sum_{j=1}^{k-1} (j\lambda + x + 1 - j) P_{x+1-j} + k(\lambda - x - 2) P_{x-k+1}. \tag{2.6}$$

Setting $P_{x+1-j} = \Delta_{x+1-j} + P_{x-j}$ in (2.6) we obtain

$$\begin{aligned} \frac{(x+1)(x+2)}{\lambda} \Delta_{x+2} = \sum_{j=1}^{k-1} & \left(j(x+1) + \frac{(\lambda - 1)j(j+1)}{2} \right) \Delta_{x+1-j} \\ & + \left(\frac{(\lambda-1)k(k+1)}{2} - 1 - x \right) P_{x+1-k}. \end{aligned} \tag{2.7}$$

Since $\lambda > 1$, we have $\Delta_0 = e^{-k\lambda} > 0$ and $\Delta_1 = (\lambda - 1)e^{-k\lambda} > 0$. An easy recursion using (2.7) shows that $\Delta_x > 0$ for $2 \le x + 2 \le \frac{(\lambda-1)k(k+1)}{2} + 1$ also. This completes the proof of the theorem. □

Proof of Theorem 2.2. For $k = 2$, Theorem 2.1 reduces to $3\lambda - 2 \le m_{2,\lambda} \le 3\lambda$. Therefore, in order to show that $m_{2,\lambda} = 3\lambda - 1$, it suffices to show that $\Delta_{3\lambda-1} > 0$ and $\Delta_{3\lambda} < 0$. However, by (2.3), $3\Delta_{3\lambda} = -2\Delta_{3\lambda-1}$. Therefore, we will only show $\Delta_{3\lambda-1} > 0$. For $\lambda = 1$, $\Delta_{3\lambda-1} = \Delta_2 = e^{-2}/2 > 0$; for $\lambda = 2$, $\Delta_{3\lambda-1} = \Delta_5 = \frac{4e^{-4}}{15} > 0$. Let $\lambda \ge 3$ and $x = 3\lambda - 3$. Using (2.6) we have

$$\tfrac{(3\lambda-1)(3\lambda-2)}{\lambda} \Delta_{3\lambda-1} = (4\lambda - 3) P_{3\lambda-3} - (4\lambda - 2) P_{3\lambda-4} = (4\lambda - 3)\Delta_{3\lambda-3} - P_{3\lambda-4}.$$

By (2.3),

$$\tfrac{1}{\lambda^3}\prod_{j=1}^{6}(6\lambda-j)\Delta_{3\lambda-1}=(64\lambda^3-267\lambda^2+360\lambda-156)\Delta_{3\lambda-7}+3(\lambda^2+8\lambda-12)P_{3\lambda-8}.$$

Therefore, $\Delta_{3\lambda-1}$ is positive since $\Delta_{3\lambda-7}>0$ by Theorem 2.1, $P_{3\lambda-8}>0$ by (1.1), and both $64\lambda^3-267\lambda^2+360\lambda-156$ and $\lambda^2+8\lambda-12$ take positive values.

For $k=3$, Theorem 2.1 reduces to $6\lambda-5\le m_{3,\lambda}\le 6\lambda$. Therefore, in order to show that $m_{3,\lambda}=6\lambda-1$, it suffices to show that $\Delta_{6\lambda-j}>0$ $(1\le j\le 4)$ and $\Delta_{6\lambda}<0$. However, $6\Delta_{6\lambda}=-5\Delta_{6\lambda-1}-3\Delta_{6\lambda-2}$ because of equation (2.3). We will show then only that $\Delta_{6\lambda-4}>0$ (the other three can be treated similarly). For $\lambda=1$, $\Delta_{6\lambda-4}=\Delta_2=e^{-3}/2>0$. Let $\lambda\ge 2$ and $x=6\lambda-6$. Using (2.6) we have

$$\begin{aligned}\frac{(6\lambda-4)(6\lambda-5)}{\lambda}\Delta_{6\lambda-4}&=(7\lambda-6)P_{6\lambda-6}+(8\lambda-7)P_{6\lambda-7}-(15\lambda-12)P_{6\lambda-8}\\&=(7\lambda-6)\Delta_{6\lambda-6}+(15\lambda-13)\Delta_{6\lambda-7}-P_{6\lambda-8}.\end{aligned}$$

By (2.3),

$$\begin{aligned}\tfrac{1}{\lambda^3}\prod_{j=4}^{8}(6\lambda-j)\Delta_{6\lambda-4}&=(1015\lambda^3-3234\lambda^2+3396\lambda-1176)\Delta_{6\lambda-9}\\&+(1203\lambda^3-3610\lambda^2+3576\lambda-1176)\Delta_{6\lambda-10}+2(199\lambda^2-372\lambda+168)P_{6\lambda-11},\end{aligned}$$

which is positive, since $\Delta_{6\lambda-9}>0$ and $\Delta_{6\lambda-10}>0$ by Theorem 2.1, $P_{6\lambda-11}>0$ by equation (1.1), and their polynomial coefficients take positive values.

When $k=4$ $(k=5)$ we use the same procedure as above to show that $\Delta_{10\lambda-j}>0$ $(2\le j\le 8)$ and $\Delta_{10\lambda-1}<0$ $(\Delta_{15\lambda-j}>0$ $(2\le j\le 13)$ and $\Delta_{15\lambda-1}<0)$. Therefore, $m_{4,\lambda}=10\lambda-2$ $(m_{5,\lambda}=15\lambda-2)$.

Remark 2.1. As k increases the computations become increasingly difficult and lengthy. We have used the computer algebra system Derive and a personal computer to check them.

Remark 2.2. According to the conjecture of Philippou and Saghafi [14], $m_{6,2}=39$. However, by equation (2.3) (and equation (1.1)), we presently find that $f_6(40;2)=0.0297464817>0.0297385179=f_6(39;2)$. Therefore the conjecture is not true at least for $k=6$ and $\lambda=2$. □

3. Further Research

In this note we have derived an upper and a lower bound for the mode(s) of the Poisson distribution of order k. Our lower bound is better than that of Luo [6]. We have also established the conjecture of Philippou and Saghafi [14] for $2\le k\le 5$ and $\lambda\in\mathbb{N}$, partially solving the open problem of Philippou [7, 8, 11]. However, the problem remains open for all other cases.

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