EXPRESSIONS FOR THE PRODUCTS OF THE SECOND ORDER LINEAR RECURRENCES

KANTAPHON KUHAPATANAKUL AND RATTANAPOL WASUTHARAT

Abstract. In this paper we consider the second order linear recurrent sequences and derive explicit formulas for the products of elements in these sequences which extend explicit formulas for the squared generalized Fibonacci numbers and the products of consecutive generalized Fibonacci numbers.

1. Introduction

The second order linear recurrence $W_n = W_n(p, q; a, b)$ is defined for $n > 0$ by

$$W_{n+1} = aW_n + bW_{n-1},$$

in which $W_0 = p$ and $W_1 = q$, where a, b, p, q are arbitrary real numbers.

As some special cases of $\{W_n\}$, define the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences as $U_n = W_n(0, 1; a, b)$ and $V_n = W_n(2, a; a, b)$, respectively. If $a = b = 1$, then $U_n = F_n$ and $V_n = L_n$ are well-known Fibonacci and Lucas numbers, respectively.

It is well-known that explicit formulas for the generalized Fibonacci and Lucas numbers are

$$U_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} a^{n-2i} b^i,$$

$$V_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} a^{n-2i} b^i,$$

respectively, see equations (2.7) and (2.8) in [2].

Recently the author [1] found explicit formulas for the squares of generalized Fibonacci numbers and the products of consecutive generalized Fibonacci numbers.

$$U_{n+1}^2 = \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j}{i} a^{2(n-i-j)} b^{i+j},$$

$$U_n U_{n+1} = \sum_{i=0}^{\lfloor (2n-1)/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j-1}{i} a^{2(n-i-j)-1} b^{i+j}.$$

The objectives here are to derive formulas for the numbers $W_n U_n$ and $W_n U_{n+1}$ which generalize the above two identities.

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2. Main Results

Definition 2.1. *Let n, i be integers with $n \geq 0$. The number $B(n,i)$ is defined as $B(0,0) = p$ and, for $n > 0$,*

$$B(n,i) = \begin{cases} qa^{n-1}, & i = 0; \\ pa^i b^i, & i = n; \\ aB(n-1,i) + ab(n-1,i-1) + b^2 B(n-2,i-1), & 0 < i < n; \\ 0, & \text{otherwise}. \end{cases}$$

We give an alternative definition of $B(n,i)$ in the binomial sums.

Theorem 2.2. *Let $n \in \mathbb{N}$ and i be non-negative integers with $i < n$. We have*

$$B(n,i) = \sum_{j=0}^{i} \frac{ipa + (n-i-j)q}{n-j} \binom{i}{j} \binom{n-j}{i} a^{n-2j-1} b^{i+j}. \tag{2.1}$$

Proof. For $n = 1$, we see that equation (2.1) holds for $i = 0$. Assume equation (2.1) is true for $n \geq 1$ and $0 \leq i < n$. By Definition 2.1 and the inductive hypothesis, we obtain

$$\begin{aligned} B(n+1,i) &= \sum_{j=0}^{i} \tfrac{ipa+(n-i-j)q}{n-j} \tbinom{i}{j} \tbinom{n-j}{i} a^{n-2j} b^{i+j} \\ &\quad + \sum_{j=0}^{i-1} \tfrac{(i-1)pa+(n-i-j+1)q}{n-j} \tbinom{i-1}{j} \tbinom{n-j}{i-1} a^{n-2j} b^{i+j} \\ &\quad + \sum_{j=0}^{i-1} \tfrac{(i-1)pa+(n-i-j)q}{n-j-1} \tbinom{i-1}{j} \tbinom{n-j-1}{i-1} a^{n-2j-2} b^{i+j+1} \\ &= \tfrac{ipa+(n-i)q}{n} \tbinom{n}{i} a^n b^i + \sum_{j=1}^{i-1} \tfrac{ipa+(n-i-j)q}{n-j} \tbinom{i}{j} \tbinom{n-j}{i} a^{n-2j} b^{i+j} + \tfrac{ipa+(n-2i)q}{n-i} \tbinom{n-i}{i} a^{n-2i} b^{2i} \\ &\quad + \tfrac{(i-1)pa+(n-i+1)q}{n} \tbinom{n}{i-1} a^n b^i + \sum_{j=1}^{i-1} \tfrac{(i-1)pa+(n-i-j+1)q}{n-j} \tbinom{i-1}{j} \tbinom{n-j}{i-1} a^{n-2j} b^{i+j} \\ &\quad + \sum_{j=1}^{i-1} \tfrac{(i-1)pa+(n-i-j+1)q}{n-j} \tbinom{i-1}{j-1} \tbinom{n-j}{i-1} a^{n-2j} b^{i+j} + \tfrac{(i-1)pa+(n-2i+1)q}{n-i} \tbinom{n-i}{i-1} a^{n-2i} b^{2i}. \end{aligned}$$

From Pascal's identity $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$ and the definition of binomial coefficient, we obtain

$$\frac{i}{n}\binom{n}{i} + \frac{i-1}{n}\binom{n}{i-1} = \frac{i}{n}\binom{n+1}{i} - \frac{1}{n}\binom{n}{i-1} = \frac{i}{n+1}\binom{n+1}{i},$$

so

$$\begin{aligned}B(n+1,i) &= \tfrac{ipa+(n-i+1)q}{n+1}\tbinom{n+1}{i}a^n b^i + \sum_{j=1}^{i-1}\tfrac{ipa+(n-i-j+1)q}{n-j+1}\tbinom{i}{j}\tbinom{n-j+1}{i}a^{n-2j}b^{i+j} \\ &\quad + \tfrac{ipa+(n-2i+1)q}{n-i+1}\tbinom{n-i+1}{i}a^{n-2i}b^{2i} \\ &= \sum_{j=0}^{i}\tfrac{ipa+(n-i-j+1)q}{n-j+1}\tbinom{i}{j}\tbinom{n-j+1}{i}a^{n-2j}b^{i+j}.\end{aligned}$$

Thus, equation (2.1) holds for $n+1$, thereby proving the theorem. □

Notes: (1) If we take $p=0$ and $q=1$ in identity (2.1) of Theorem 2.2, then $B(n,i)$ is exactly $T(n+1,i)$ of Definition 3.1 in [1].
(2) We see that all the terms in the summation of $B(n,i)$ are zero when $j>\min\{i,n-i\}$. For completeness, we write $B(n,i)$, identity (2.1), as

$$B(n,i) = \sum_{j=0}^{\lfloor\frac{n+i}{3}\rfloor}\frac{ipa+(n-i-j)q}{n-j}\binom{i}{j}\binom{n-j}{i}a^{n-2j-1}b^{i+j}. \tag{2.2}$$

Now, we state explicit formulas for the numbers $U_{n+1}W_n$ and U_nW_n. Their proofs will be given in the next section.

Theorem 2.3. *For any positive integer n, we have*

(i) $U_{n+1}W_n = \displaystyle\sum_{i=0}^{\lfloor\frac{2n}{3}\rfloor} B(2n-2i,i)$.

(ii) $U_nW_n = \displaystyle\sum_{i=0}^{\lfloor\frac{2n-1}{3}\rfloor} B(2n-2i-1,i)$.

By using identity (2.2), we obtain the following corollary.

Corollary 2.4. *For any positive integer n, we have*

(i) $U_{n+1}W_n = \displaystyle\sum_{i=0}^{\lfloor\frac{2n}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{2n-i}{3}\rfloor}\frac{ipa+(2n-3i-j)q}{2n-2i-j}\binom{i}{j}\binom{2n-2i-j}{i}a^{2(n-i-j)-1}b^{i+j}$.

(ii) $U_nW_n = \displaystyle\sum_{i=0}^{\lfloor\frac{2n-1}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{2n-i-1}{3}\rfloor}\frac{ipa+(2n-3i-j-1)q}{2n-2i-j-1}\binom{i}{j}\binom{2n-2i-j-1}{i}a^{2(n-i-j-1)}b^{i+j}$.

Corollary 2.5. *For any positive integer n, we have*

(1) $U_{n+1}U_n = \displaystyle\sum_{i=0}^{\lfloor\frac{2n}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{2n-i}{3}\rfloor}\binom{i}{j}\binom{2n-2i-j-1}{i}a^{2(n-i-j)-1}b^{i+j}$.

(2) $U_n^2 = \displaystyle\sum_{i=0}^{\lfloor\frac{2n-1}{3}\rfloor}\sum_{j=0}^{\lfloor\frac{2n-i-1}{3}\rfloor}\binom{i}{j}\binom{2n-2i-j-2}{i}a^{2(n-i-j-1)}b^{i+j}$.

(3) $U_{n+1}V_n = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{2n-i-j}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i} a^{2(n-i-j)} b^{i+j}.$

(4) $U_nV_n = \sum_{i=0}^{\lfloor \frac{2n-1}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i-1}{3} \rfloor} \frac{2n-i-j-1}{2n-2i-j-1} \binom{i}{j} \binom{2n-2i-j-1}{i} a^{2(n-i-j)-1} b^{i+j}.$

(5) $V_n^2 - b^2U_{n-1}^2 = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{2n-j}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i} a^{2n-2i-2j} b^{i+j}.$

(6) $U_{n-1}U_{n+1} = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{i}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i} a^{2n-2i-2j} b^{i+j-1}.$

Proof. We write $w_n = W_n(3, a; a, b)$. The following identities are easily verified

$$V_n^2 - b^2U_{n-1}^2 = U_{n+1}w_n. \tag{2.3}$$

$$2bU_{n-1}U_{n+1} = U_{n+1}w_n - U_{n+1}^2. \tag{2.4}$$

For parts (1)–(4), simply apply Corollary 2.4 with $W_n = U_n$ or V_n. Taking $p = 3$ in Corollary 2.4(i) and using identities (2.3) and (2.4), we obtain parts (5)–(6). □

Identities (1) and (2) of Corollary 2.5 are the two identities which appeared in Section 1, i.e. see [1]. Specializing certain parameters in Corollary 2.5, several identities follow easily as we illustrate now. Taking $U_n = F_n$ and $V_n = L_n$, we can rewrite as

(1) $F_{n+1}F_n = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \binom{i}{j} \binom{2n-2i-j-1}{i}.$

(2) $F_n^2 = \sum_{i=0}^{\lfloor \frac{2n-1}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i-1}{3} \rfloor} \binom{i}{j} \binom{2n-2i-j-2}{i}.$

(3) $F_{n+1}L_n = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{2n-i-j}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i}.$

(4) $F_nL_n = \sum_{i=0}^{\lfloor \frac{2n-1}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i-1}{3} \rfloor} \frac{2n-i-j-1}{2n-2i-j-1} \binom{i}{j} \binom{2n-2i-j-1}{i}.$

(5) $L_n^2 - F_{n-1}^2 = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{2n-j}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i}.$

(6) $F_{n-1}F_{n+1} = \sum_{i=0}^{\lfloor \frac{2n}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{2n-i}{3} \rfloor} \frac{i}{2n-2i-j} \binom{i}{j} \binom{2n-2i-j}{i}.$

3. Proof of Theorem 2.3

We first provide an identity which extends the identities of Lemma 4.1 in [1].

Lemma 3.1. *Let $n \in \mathbb{N}$ and k be non-negative integer. Then*

$$U_{n+k+2}W_{n+2} = (a^2+b)U_{n+k+1}W_{n+1} + (a^2b+b^2)U_{n+k}W_n - b^3U_{n+k-1}W_{n-1}.$$

Proof. By the definition of U_n and W_n, we obtain

$$\begin{aligned} U_{n+k+2}W_{n+2} &= (aU_{n+k+1} + bU_{n+k})(aW_{n+1} + bW_n) \\ &= a^2U_{n+k+1}W_{n+1} + b^2U_{n+k}W_n + abU_{n+k+1}W_n + abU_{n+k}W_{n+1} \\ &= a^2U_{n+k+1}W_{n+1} + b^2U_{n+k}W_n + abW_n(aU_{n+k} + bU_{n+k-1}) \\ &\quad + bW_{n+1}(U_{n+k+1} - bU_{n+k-1}) \\ &= (a^2+b)U_{n+k+1}W_{n+1} + (a^2b+b^2)U_{n+k}W_n - b^3U_{n+k-1}W_{n-1}, \end{aligned}$$

as desired. □

Proof of Theorem 2.3. Since the proofs of both part (i) and part (ii) are quite similar, we only give a proof for part (i). By induction on n, we see that identity (i) holds for $n = 1, 2, 3$. Now assume identity (i) is true for all integers $n \geq 1$. By Lemma 3.1 and the inductive hypothesis, we obtain

$$\begin{aligned} U_{n+2}W_{n+1} &= (a^2+b)U_{n+1}W_n + (a^2b+b^2)U_nW_{n-1} - b^3U_{n-1}W_{n-2} \\ &= (a^2+b)\sum_{i\geq 0} B(2n-2i, i) + (a^2b+b^2)\sum_{i\geq 0} B(2n-2i-2, i) \\ &\quad - b^3\sum_{i\geq 0} B(2n-2i-4, i) \\ &= a^2B(2n,0) + bB(2n,0) + (a^2+b)\sum_{i\geq 1} B(2n-2i, i) \\ &\quad + (a^2b+b^2)\sum_{i\geq 1} B(2n-2i, i-1) - b^3\sum_{i\geq 1} B(2n-2i-2, i-1) \\ &= B(2n+2,0) + a^2\sum_{i\geq 1} B(2n-2i, i) + bB(2n,0) + ab\sum_{i\geq 1} B(2n-2i-1, i) \\ &\quad + ab^2\sum_{i\geq 1} B(2n-2i-1, i-1) + b^3\sum_{i\geq 1} B(2n-2i-2, i-1) \\ &\quad + (a^2b+b^2)\sum_{i\geq 1} B(2n-2i, i-1) - b^3\sum_{i\geq 1} B(2n-2i-2, i-1) \\ &= B(2n+2,0) + a\sum_{i\geq 1} B(2n-2i+1, i) + ab\sum_{i\geq 1} B(2n-2i+1, i-1) \\ &\quad + b^2\sum_{i\geq 1} B(2n-2i, i-1) \\ &= \sum_{i\geq 0} B(2n-2i+2, i). \end{aligned}$$

Therefore, the result is true for every n.

References

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Department of Mathematics, Kasetsart University, Bangkok 10900, Thailand
E-mail address: fscikpkk@ku.ac.th

Department of Mathematics, Kasetsart University, Bangkok 10900, Thailand
E-mail address: nung31@hotmail.com