COMBINATORIAL PROOFS OF DETERMINANT FORMULAS FOR THE FIBONACCI AND LUCAS POLYNOMIALS

MARK SHATTUCK

Abstract. We provide combinatorial proofs of several formulas for the Fibonacci and Lucas polynomials as determinants of some recently introduced Hessenberg matrices. Our arguments make use of the basic definition of the determinant as a signed sum over the symmetric group and generalize prior results, which were shown algebraically using cofactor expansion and recurrences.

1. Introduction

Let $F_n(a,b)$ be the nth Fibonacci polynomial defined by $F_n(a,b) = aF_{n-1}(a,b) + bF_{n-2}(a,b)$ if $n \geq 2$, with $F_0(a,b) = 0$ and $F_1(a,b) = 1$, where a and b are indeterminates. Let $L_n(a,b)$ be the Lucas polynomial defined by $L_n(a,b) = aL_{n-1}(a,b) + bL_{n-2}(a,b)$ if $n \geq 2$, with $L_0(a,b) = 2$ and $L_1(a,b) = a$. The Fibonacci and Lucas polynomials are also given by the well-known formulas

$$F_n(a,b) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} a^{n-1-2k} b^k, \qquad n \geq 1,$$

and

$$L_n(a,b) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} a^{n-2k} b^k, \qquad n \geq 1.$$

Note that the $F_n(a,b)$ and $L_n(a,b)$ reduce to the Fibonacci and Lucas sequences F_n and L_n, respectively, when $a = b = 1$; see, respectively, sequences A000045 and A000032 in [10].

Given an $n \times n$ matrix A, with entries a_{ij}, the *determinant of A*, denoted $|A|$, is defined by

$$|A| = \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}, \tag{1.1}$$

where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$.

A matrix is said to be (lower) Hessenberg if all of its entries above the superdiagonal are zero. The Hessenberg matrix [6]

$$A_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ & & & & 1 \\ 1 & \cdots & & 1 & 2 \end{pmatrix}_{n \times n} \tag{1.2}$$

has as its determinant F_{n+2}. Recently, several Hessenberg matrices have been introduced whose determinants are $F_n(a,1)$ and $L_n(a,1)$, respectively, see [8] and [9]. These determinants were obtained by algebraic methods using cofactor expansions, recurrences and properties of the determinant.

Here, we interpret several of these determinants, combinatorially, as signed sums over various classes of compositions. In addition to imparting a more visceral understanding to why an algebraic expression assumes a particular value, combinatorial proofs, perhaps more importantly, often allow for further generalizations upon consideration of additional parameters. We provide here some examples of this technique and generalize, through combinatorial arguments, the results in [8] and [9] to obtain determinant expressions for $F_n(a,b)$ and $L_n(a,b)$ as well as additional separate expressions for the Fibonacci and Lucas polynomials of even and odd index. We remark that combinatorial proofs have recently been given for other determinants, including those of Vandermonde's matrix [3], of matrices whose entries are the Fibonacci [1] and Catalan [2] numbers, and of other matrices related to a class of Fibonacci-type numbers [5].

By a *composition* of a positive integer n, we will mean a sequence of positive integers, called *parts*, whose sum is n. Given a composition of n, we will call a part j *initial* if it is the left-most part of the composition and will call it *terminal* if it is the right-most part. Let $\mathcal{F}_n$ denote the set of compositions of n with parts belonging to $\{1,2\}$ and let $\mathcal{L}_n$ denote the same set of compositions, but where a 2 starting a composition may be marked (which we'll indicate by underscoring). For example, the compositions $(2,1,2)$ and $(1,2,2)$ both belong to $\mathcal{F}_5$ and $\mathcal{L}_5$, while $(\underline{2},2,1) \in \mathcal{L}_5 - \mathcal{F}_5$. Members of $\mathcal{F}_n$ or $\mathcal{L}_n$ may be regarded, equivalently, as square-and-domino tilings which cover the numbers $1,2,\ldots,n$, written either in a straight line or around a circle, respectively, see [4, Chapters 1 and 2].

Given a composition λ with parts in $\{1,2\}$, let us define the weight of λ by

$$\nu(\lambda) = a^{\nu_1(\lambda)} b^{\nu_2(\lambda)},$$

where $\nu_1(\lambda)$ and $\nu_2(\lambda)$ denote the number of parts of λ of sizes 1 and 2, respectively (in the case that $\lambda \in \mathcal{L}_n - \mathcal{F}_n$, the initial marked 2 is also counted by ν_2). Recall that

$$F_{n+1}(a,b) = \sum_{\lambda \in \mathcal{F}_n} \nu(\lambda) \qquad \text{and} \qquad L_n(a,b) = \sum_{\lambda \in \mathcal{L}_n} \nu(\lambda), \tag{1.3}$$

see, e.g., [4], which is easy to show using the defining recurrences. In particular, taking $a = b = 1$, we get $|\mathcal{F}_n| = F_{n+1}$ and $|\mathcal{L}_n| = L_n$; see, e.g., [11, p. 46] and [4, p. 18].

Note that in the case of a Hessenberg matrix A, the term corresponding to $\pi \in \mathcal{S}_n$ in the sum on the right-hand side of (1.1) is non-zero only if each cycle of π is of the form $(i, i+1, \ldots, i+d)$ for some i and d. Note that such permutations are clearly synonymous with compositions of n, upon ordering the cycles in some manner and identifying cycle lengths as parts. In what follows, we will denote the subset of $\mathcal{S}_n$ consisting of these permutations by $\mathcal{T}_n$, which we'll also use to denote the set of compositions of n.

2. Combinatorial Proofs

We first consider two classes of Hessenberg matrices whose determinants may be expressed in terms of the Fibonacci and Lucas polynomials. If a and b are indeterminates, then let $A_n = A_n(a,b)$ denote the $n \times n$ Hessenberg matrix in which all the entries below the main diagonal and all superdiagonal entries are b and all entries along the main diagonal are $a^2 + b$. Let $B_n = B_n(a,b)$ be the same as A_n except that the first row, first column entry is $a^2 + 3b$. Algebraic proofs have been given establishing the determinants of $|A_n|$ and $|B_n|$ in the case when $b = 1$; see [8] and [9], respectively. The matrices A_4 and B_4 are shown below.

$$A_4 = \begin{pmatrix} a^2+b & b & 0 & 0 \\ b & a^2+b & b & 0 \\ b & b & a^2+b & b \\ b & b & b & a^2+b \end{pmatrix},$$

$$B_4 = \begin{pmatrix} a^2+3b & b & 0 & 0 \\ b & a^2+b & b & 0 \\ b & b & a^2+b & b \\ b & b & b & a^2+b \end{pmatrix}.$$

Let $\mathcal{U}_n$ denote the set of compositions of n, where parts of size 1 may be marked. Given $\lambda \in \mathcal{U}_n$, let $\alpha(\lambda)$ denote the number of marked parts of size 1 and let $\beta(\lambda)$ denote the number of even parts. Define the (signed) weight of λ, which we'll denote $w(\lambda)$, by

$$w(\lambda) = (-1)^{\beta(\lambda)} a^{2\alpha(\lambda)} b^{n-\alpha(\lambda)},$$

where a and b are indeterminates. In our evaluations of the determinants $|A_n|$ and $|B_n|$, we will need the following two lemmas.

Lemma 2.1. *If $n \geq 1$, then*

$$|A_n| = \sum_{\lambda \in \mathcal{U}_n} w(\lambda). \tag{2.1}$$

Proof. By (1.1), showing (2.1) is equivalent to showing

$$\sum_{\pi \in \mathcal{T}_n} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} = \sum_{\lambda \in \mathcal{U}_n} w(\lambda), \tag{2.2}$$

where a_{ij} denotes the (i,j)th entry of A_n. For (2.2), first note that each cycle of length $m \geq 2$ within $\pi \in \mathcal{T}_n$ contributes a factor of b^m towards the product term corresponding to the permutation π in the sum on the right-hand side. If a cycle has length one, then it contributes a^2+b towards this product. This may be thought of as differentiating 1-cycles into two types by marking some subset of them and leaving all other cycles of π unmarked. Then unmarked cycles contribute weight b^m, where m denotes the length of the cycle, with marked 1-cycles each contributing a^2. Note that this yields a member of $\mathcal{U}_n$ of the same weight, which we'll denote by λ, upon identifying cycle lengths of π as parts (where π is understood to have been written as a product of disjoint cycles arranged in increasing order of smallest elements).

We now show that π and λ have the same sign. Let $num(\pi)$ be the number of cycles of π and let $num_o(\pi)$ and $num_e(\pi)$ be the number of cycles of π of odd and even length, respectively. Then the sign of λ is the same as $\operatorname{sgn}(\pi) = (-1)^{n-num(\pi)}$ since $num(\pi) - num_e(\pi) = num_o(\pi) \equiv n \pmod 2$ implies $\beta(\lambda) = num_e(\pi) \equiv n - num(\pi) \pmod 2$, which completes the proof of (2.2). $\square$

Let $\mathcal{U}_n^*$ denote the subset of $\mathcal{U}_n$ whose members contain only marked and unmarked 1's as parts and in which no two unmarked 1's are directly adjacent.

Lemma 2.2. *If $n \geq 1$, then*

$$a^{n-1} F_{n+2}(a,b) = \sum_{\lambda \in \mathcal{U}_n^*} w(\lambda). \tag{2.3}$$

Proof. Note first that members of $\mathcal{U}_n^*$ have positive sign as they contain no parts of even length. Let us first add a marked 1 to the end of each $\lambda \in \mathcal{U}_n^*$. We may then identify λ as a member of $\mathcal{F}_{n+1}$, upon regarding each occurrence of an unmarked 1 directly followed by a marked 1 as a "2" and regarding each remaining marked 1 as a "1". Note that within this member of $\mathcal{F}_{n+1}$, all 1's would have weight a^2, except for a terminal 1, which would have unit weight, and all 2's would have weight a^2b, except for a terminal 2, which would have weight b. Then there are $n-1$ additional factors of a which arise when comparing our weighting for a member of $\mathcal{F}_{n+1}$ with the usual one in (1.3). This is seen upon considering, separately, the cases when a member of $\mathcal{F}_{n+1}$ ends in a 1 or in a 2. Thus, under the present weighting, the total weight of all the members of $\mathcal{F}_{n+1}$, and hence, of $\mathcal{U}_n^*$, is given by $a^{n-1}F_{n+2}(a,b)$, which completes the proof. □

We now evaluate the determinants of the matrices A_n and B_n.

Proposition 2.3. *If $n \geq 1$, then*

$$|A_n| = a^{n-1}F_{n+2}(a,b) \text{ and } |B_n| = a^{n-2}L_{n+2}(a,b).$$

Proof. By Lemma 2.1, the first statement is equivalent to

$$\sum_{\lambda \in \mathcal{U}_n} w(\lambda) = a^{n-1}F_{n+2}(a,b), \quad n \geq 1. \tag{2.4}$$

To prove (2.4), it is enough to define a sign-changing, weight-preserving involution of $\mathcal{U}_n - \mathcal{U}_n^*$, by Lemma 2.2. We define the involution as follows. If $n \geq 2$, then consider the subset of $\mathcal{U}_n$ whose members contain no marked 1's, which we'll also denote by $\mathcal{T}_n$. Note that $\mathcal{T}_n \subseteq \mathcal{U}_n - \mathcal{U}_n^*$. We first define an involution of $\mathcal{T}_n$ which we will extend to $\mathcal{U}_n - \mathcal{U}_n^*$. To do so, we'll define separate involutions on $\mathcal{T}_n - \mathcal{T}_n'$ and on $\mathcal{T}_n'$, where $\mathcal{T}_n' \subseteq \mathcal{T}_n$ comprises those members which contain no odd parts greater than or equal 3 and in which no part of size 1 is directly followed by an even part.

To define the first involution, suppose $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{T}_n - \mathcal{T}_n'$. Let i_o denote the smallest index i such that one of the following conditions holds:

$$\text{(i) } \lambda_i \geq 3 \text{ is odd} \qquad \text{or} \qquad \text{(ii) } \lambda_i = 1 \text{ and } \lambda_{i+1} \text{ is even.}$$

Let $\widehat{\lambda}$ denote the member of $\mathcal{T}_n - \mathcal{T}_n'$ obtained by replacing λ_{i_o} with the parts $1, \lambda_{i_o} - 1$ if (i) occurs and replacing $1, \lambda_{i_o+1}$ with the single part $1 + \lambda_{i_o+1}$ if (ii) occurs. Then $w(\lambda) = -w(\widehat{\lambda})$ and the mapping $\lambda \mapsto \widehat{\lambda}$ is an involution of $\mathcal{T}_n - \mathcal{T}_n'$.

We now define an involution of $\mathcal{T}_n'$. First note that if n is even, then members of $\mathcal{T}_n'$ either end in an even part or in a sequence of consecutive 1's of even length. Then the mapping $\lambda \mapsto \bar{\lambda}$ obtained by replacing one of these options with the other defines a sign-changing involution of $\mathcal{T}_n'$. If n is odd, then all members of T_n' have final part 1 and we apply the mapping just described to the compositions of $n-1$ obtained by ignoring this 1, which are clearly synonymous with the members of $\mathcal{T}_{n-1}'$. Combining the two mappings yields the desired involution of $\mathcal{T}_n$. See Figure 1 below.

$$\alpha = 1\ 1\ \mathbf{5}\ 4\ 1\ 3\ 1\ 3 \ \rightarrow\ \widehat{\alpha} = 1\ 1\ \mathbf{1}\ \mathbf{4}\ 4\ 1\ 3\ 1\ 3$$
$$\beta = 4\ 2\ 6\ 2\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\ 1 \ \rightarrow\ \bar{\beta} = 4\ 2\ 6\ 2\ \mathbf{4}\ 1$$

FIGURE 1. Examples of the mappings $\lambda \mapsto \widehat{\lambda}$ and $\lambda \mapsto \bar{\lambda}$ when $n = 19$.

We now extend this involution to $\mathcal{U}_n - \mathcal{U}_n^*$. We first express $\lambda \in \mathcal{U}_n - \mathcal{U}_n^*$ as

$$\lambda = \alpha_1 \underline{1} \alpha_2 \underline{1} \cdots \alpha_r \underline{1} \alpha_{r+1},$$

where marked 1's are indicated by underscoring and each α_i is a (possibly empty) composition containing no marked 1's. Let j_o denote the smallest index j such that $|\alpha_j| \geq 2$; note that j_o exists since $\lambda \in \mathcal{U}_n - \mathcal{U}_n^*$. Let $\widetilde{\lambda}$ be the composition obtained by replacing α_{j_o} with α'_{j_o}, where the prime denotes the (composite) mapping defined above on $\mathcal{T}_n$. Then the mapping $\lambda \mapsto \widetilde{\lambda}$ is an involution of $\mathcal{U}_n - \mathcal{U}_n^*$, with $w(\lambda) = -w(\widetilde{\lambda})$, which completes the proof of (2.4). See Figure 2 below.

To evaluate $|B_n|$, we consider the same set of compositions $\mathcal{U}_n$, except that now an unmarked 1 starting a composition can come in one of two colors or be left uncolored (any unmarked 1 contributes b towards the weight of a tiling). If such a composition does not start with a colored 1, then the prior proof shows that there are $a^{n-1}F_{n+2}(a,b)$ possibilities in this case. If it does start with a colored 1, then there are $2b \cdot a^{n-2}F_{n+1}(a,b)$ possibilities, by the prior proof applied to compositions of size $n-1$ obtained by deleting the first part. Thus, we have

$$\begin{aligned}|B_n| &= a^{n-1}F_{n+2}(a,b) + 2a^{n-2}bF_{n+1}(a,b) = a^{n-2}(aF_{n+2}(a,b) + 2bF_{n+1}(a,b)) \\ &= a^{n-2}L_{n+2}(a,b),\end{aligned}$$

by the well-known relation $L_m(a,b) = aF_m(a,b) + 2bF_{m-1}(a,b)$, which is easily realized combinatorially. □

$$\alpha = 1\ \underline{1}\ \underline{1}\ 1\ \underline{1}\ 1\ \mathbf{3}\ 2\ 1\ \underline{1}\ 2\ 4 \ \rightarrow\ \widetilde{\alpha} = 1\ \underline{1}\ \underline{1}\ 1\ \underline{1}\ 1\ \mathbf{1}\ \mathbf{2}\ 2\ 1\ \underline{1}\ 2\ 4$$

$$\beta = 1\ \underline{1}\ \underline{1}\ 4\ \mathbf{2}\ \underline{1}\ 4\ \underline{1}\ 2\ 1\ 1 \ \rightarrow\ \widetilde{\beta} = 1\ \underline{1}\ \underline{1}\ 4\ \mathbf{1}\ \mathbf{1}\ \underline{1}\ 4\ \underline{1}\ 2\ 1\ 1$$

FIGURE 2. Examples of the mapping $\lambda \mapsto \widetilde{\lambda}$ when $n = 19$.

Example 2.4. *Letting $a = b = 1$ in Proposition 2.3 implies $|A_n(1,1)| = F_{n+2}$ and $|B_n(1,1)| = L_{n+2}$. Letting $a = 2$ and $b = 1$ shows that the $n \times n$ Hessenberg matrix with 1's on the superdiagonal and below the main diagonal and 5's on the main diagonal has determinant $2^{n-1}P_{n+2}$, where P_n denotes the Pell number sequence* (*see* [10, A000129]) *given by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ if $n \geq 2$, with $P_0 = 0$ and $P_1 = 1$.*

Remark 2.1. *Let A'_n be the matrix obtained by replacing a^2+b with b in the last main diagonal entry of A_n. Then reasoning as in the proof of Proposition 2.3 shows that $|A'_n|$ gives the total weight of all members of $\mathcal{U}_n^*$ which end in an unmarked 1. Since no two unmarked 1's can be adjacent, the second-to-last part must be a marked 1, which implies that the total weight is $a^2b \cdot a^{n-3}F_n(a,b)$ and thus $|A'_n| = a^{n-1}bF_n(a,b)$, as shown in* [8] *in the case $b = 1$.*

Let us now consider the $n \times n$ Hessenberg matrix $C_n = C_n(a,b)$ having $-b$ on the superdiagonal, a^2+b on the main diagonal, and a^2 for all entries below the main diagonal. Let D_n denote the matrix whose entries are all the same as those of C_n except that $(a^2+3b, a^2+2b, \ldots, a^2+2b)$ replaces $(a^2+b, a^2, \ldots, a^2)$ for the entries of the first column. See [8] and [9], where algebraic proofs were given for the determinants of C_n and D_n in the case when $b = 1$. The matrices C_4 and D_4 are shown below.

$$C_4 = \begin{pmatrix} a^2+b & -b & 0 & 0 \\ a^2 & a^2+b & -b & 0 \\ a^2 & a^2 & a^2+b & -b \\ a^2 & a^2 & a^2 & a^2+b \end{pmatrix},$$

$$D_4 = \begin{pmatrix} a^2+3b & -b & 0 & 0 \\ a^2+2b & a^2+b & -b & 0 \\ a^2+2b & a^2 & a^2+b & -b \\ a^2+2b & a^2 & a^2 & a^2+b \end{pmatrix}.$$

Proposition 2.5. *If $n \geq 1$, then*

$$|C_n| = F_{2n+1}(a,b) \text{ and } |D_n| = \frac{1}{a} L_{2n+1}(a,b).$$

Proof. To show the first statement, note that a cycle of length $m \geq 2$ within a member π of $\mathcal{T}_n$ contributes a^2b^{m-1} towards the product in (1.1), with the sign always positive. A 1-cycle clearly contributes a^2 or b towards this product; we subsequently mark the 1-cycles that contribute b so as to differentiate them from the others.

By a "marked" composition of n, we will mean one in which some of the 1's may be marked, the set of which we denote by $\mathcal{U}_n$. Then $|C_n|$ gives the sum of the weights of all the members of $\mathcal{U}_n$, where unmarked parts of *any* size $m \geq 1$ have weight a^2b^{m-1} and marked 1's have weight b. We convert such compositions of n into members of $\mathcal{F}_{2n}$ by replacing each unmarked part $m \geq 1$ with the sequence of parts $12^{m-1}1$ and replacing each marked 1 with a 2. See Figure 3. This yields all members of $\mathcal{F}_{2n}$, weighted as in (1.3). Since this mapping between $\mathcal{U}_n$ and $\mathcal{F}_{2n}$ is seen to be a weight-preserving bijection, we have $|C_n| = F_{2n+1}(a,b)$.

$$\underline{1}\ 2\ 1\ \underline{1}\ \underline{1}\ 4\ \ \rightarrow 2\ 1\ 2\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ \ \in \mathcal{F}_{20}$$

FIGURE 3. Example of the mapping on $\mathcal{U}_n$ when $n = 10$.

To show the second statement, first let $\mathcal{D}_n$ denote the set of "marked colored" compositions of size n in which some of the 1's may be marked and an unmarked initial part of *any* size $m \geq 1$ may be colored in one of two ways (or be left uncolored). An initial colored part of size m receives weight b^m, with all other weights assigned as in the previous paragraph. Then $|D_n|$ gives the total weight of all the members of $\mathcal{D}_n$. We now convert the members of $\mathcal{D}_n$ into compositions with parts in $\{1,2\}$ as before, with the following additional rules: (i) An initial colored part $m \geq 1$ becomes the sequence of parts 2^m1, where the first 2 is unmarked if the initial part is of the first color and is marked if it is of the second; (ii) If the member of $\mathcal{D}_n$ does not start with a colored part, then write a 1 and proceed as before. This operation on $\mathcal{D}_n$ then defines a *near* weight-preserving bijection with the set $\mathcal{L}_{2n+1}$ having the usual weighting, see (1.3). Only a factor of a corresponding to the left-most part of size 1 is missing, which implies the second statement. See Figure 4, where the subscript denotes the choice of color for an initial colored part within a member of $\mathcal{D}_n$. □

$$3\ \underline{1}\ 2\ 1\ \underline{1}\ 2\ \ \rightarrow 1\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ \ \in \mathcal{L}_{21}$$
$$3_2\ \underline{1}\ 2\ 1\ \underline{1}\ 2\ \ \rightarrow \underline{2}\ 2\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ \ \in \mathcal{L}_{21}$$

FIGURE 4. Example of the mapping on $\mathcal{D}_n$ when $n = 10$.

Example 2.6. *Letting $a = b = 1$ in Proposition 2.5 implies $|C_n(1,1)| = F_{2n+1}$ and $|D_n(1,1)| = L_{2n+1}$. Letting $a = 2$ and $b = 1$ shows that the $n \times n$ Hessenberg matrix with -1's on the superdiagonal, 4's below the main diagonal, and 5's on the main diagonal has determinant P_{2n+1}.*

Next, we consider the $n \times n$ Hessenberg matrix E_n having $-b$ on the superdiagonal, a^2 for all entries below the main diagonal, and $a^2 + b$ for all entries on the main diagonal, except the first, which is a^2 (we take $E_1 = [a^2]$). Let H_n be the matrix with the same entries as E_n except that the first row, first column entry is $a^2 + 2b$ instead of a^2 (we take $H_1 = [a^2 + 2b]$). The matrices E_n and H_n and their determinants were considered in [8] and [9], respectively, in the case $b = 1$. Below are the matrices E_4 and H_4.

$$E_4 = \begin{pmatrix} a^2 & -b & 0 & 0 \\ a^2 & a^2+b & -b & 0 \\ a^2 & a^2 & a^2+b & -b \\ a^2 & a^2 & a^2 & a^2+b \end{pmatrix},$$

$$H_4 = \begin{pmatrix} a^2+2b & -b & 0 & 0 \\ a^2 & a^2+b & -b & 0 \\ a^2 & a^2 & a^2+b & -b \\ a^2 & a^2 & a^2 & a^2+b \end{pmatrix}.$$

Proposition 2.7. *If $n \geq 1$, then*

$$|E_n| = aF_{2n}(a,b) \text{ and } |H_n| = L_{2n}(a,b).$$

Proof. Reasoning as in the proof above for $|C_n|$, we see that $|E_n|$ gives the total weight of all "marked" compositions of n in which some of the 1's may be marked, but not an initial 1. These may then be converted as described above to members of $\mathcal{F}_{2n}$ which start with a 1, whose total weight is $aF_{2n}(a,b)$, which gives the first statement.

For the second statement, we again reason as in the proof above for $|C_n|$, but allow, additionally, for an initial 1 to be marked in a second way. Then $|H_n|$ gives the total weight of all such compositions of size n. We convert these compositions into compositions of size $2n$ with parts in $\{1,2\}$ as before, with the addition that an initial 1 marked in the second way becomes a marked 2. This yields a bijection now with the set $\mathcal{L}_{2n}$ and implies $|H_n| = L_{2n}(a,b)$. □

Finally, let $G_n = G_n(a,b,c,d)$ denote the generalized Fibonacci sequence defined by $G_n = aG_{n-1} + bG_{n-2}$ if $n \geq 2$, with $G_0 = c$ and $G_1 = d$. Let $J_n = J_n(a,b,c,d)$ be the $n \times n$ Hessenberg matrix, see [9], whose superdiagonal is $d, b, b, \ldots$ and whose main diagonal is $c, 0, a, a, \ldots$, with all subdiagonal entries -1 and all other entries below the main diagonal zero. The matrix J_4 is illustrated below.

$$J_4 = \begin{pmatrix} c & d & 0 & 0 \\ -1 & 0 & b & 0 \\ 0 & -1 & a & b \\ 0 & 0 & -1 & a \end{pmatrix}.$$

The following result, see [9], expresses the general sequence G_n as a determinant.

Proposition 2.8. *If $n \geq 1$, then $|J_n| = G_{n-1}$.*

Proof. If $n = 1, 2$, then the result is clear, so assume $n \geq 3$. Let $\mathcal{F}_n^* \subseteq \mathcal{F}_n$ consist of those compositions which do not start with two or more 1's. By (1.1), we see that $|J_n|$ gives the total weight of all members of $\mathcal{F}_n^*$ where individual parts are assigned weights as follows: (i) initial 1's and 2's have weights of c and d, respectively, and (ii) all other 1's and 2's have weights of a and b, respectively. Furthermore, note that, from the definition, it is seen that G_{n-1} gives the total weight of all members of $\mathcal{F}_{n-1}$ where individual parts are assigned weights as follows: (i) initial 1's and 2's have weights of d and bc, respectively, and (ii) all other 1's and 2's have weights of a and b, respectively.

To show $|J_n| = G_{n-1}$, it is enough to identify a bijection from $\mathcal{F}_n^*$ to $\mathcal{F}_{n-1}$ which respects the assigned weights above. To do so, either remove an initial 1 from $\lambda \in \mathcal{F}_n^*$ or change an initial 2 to a 1. The former yields all the members of $\mathcal{F}_{n-1}$ starting with a 2, while the latter gives all of those starting with a 1. It is seen that this mapping is a bijection from $\mathcal{F}_n^*$ to $\mathcal{F}_{n-1}$ with the weights as defined. $\square$

3. Conclusion

Here, we have provided combinatorial proofs of two sets of determinant expressions involving Hessenberg matrices for the Fibonacci and Lucas polynomials. In one case, we made use of a sign-changing involution when the related determinant sum in (1.1) contained both positive and negative terms, while in the other where the sum contained only positive terms, it was a direct enumeration. Comparable formulas involving *permanents* may also be given, upon modifying the matrices slightly, as was done in [8] and [9] when $b = 1$. Similar arguments can perhaps be given for other combinatorial determinant expressions appearing in the literature. For example, the formulas appearing in [7] express sequences satisfying general second order recurrences as determinants of tridiagonal matrices whose entries are simple functions of the golden ratio α. Perhaps these determinants too may be afforded combinatorial explanations using a suitable probabilistic interpretation for α, such as the one described in [4, Section 9.3].

4. Acknowledgement

We wish to thank the referee for a careful reading of the manuscript and for helpful suggestions which improved its presentation.

References

[1] A. T. Benjamin, N. T. Cameron, and J. J. Quinn, *Fibonacci determinants – A combinatorial approach*, The Fibonacci Quarterly, **45.1** (2007), 39–55.
[2] A. T. Benjamin, N. T. Cameron, J. J. Quinn, and C. R. Yerger, *Catalan determinants – A combinatorial approach*, Congr. Numer.: Proceedings of the Twelfth Conference on Fibonacci Numbers and their Applications **200** (2010), 27–34.
[3] A. T. Benjamin and G. P. Dresden, *A combinatorial proof of Vandermonde's determinant*, Amer. Math. Monthly, **114** (2007), 338–341.
[4] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof,* The Dolciani Mathematical Expositions, 27, Mathematical Association of America, 2003.
[5] A. T. Benjamin and M. A. Shattuck, *Recounting determinants for a class of Hessenberg matrices*, Integers, **7** (2007), #A55.
[6] M. Esmaeili, *More on the Fibonacci sequence and Hessenberg matrices*, Integers, **6** (2006), #A32.
[7] E. Kiliç and D. Tasci, *On the second order linear recurrences by tridiagonal matrices*, Ars Combin., **91** (2009), 11–18.
[8] E. Kiliç and D. Tasci, *On the generalized Fibonacci and Pell sequences by Hessenberg matrices*, Ars Combin., **94** (2010), 161–174.

[9] E. Kiliç, D. Tasci, and P. Haukkanen, *On the generalized Lucas sequences by Hessenberg matrices*, Ars Combin., **95** (2010), 383–395.
[10] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org.
[11] R. P. Stanley, *Enumerative Combinatorics, Vol. I*, Cambridge University Press, 1997.

MSC2010: 05A19, 11B39

Department of Mathematics, University of Tennessee, Knoxville, TN 37996
E-mail address: shattuck@math.utk.edu