A RESULT ABOUT CYCLES IN DUCCI SEQUENCES

CHRISTIAN AVART

ABSTRACT. We prove that for any $k \in \mathbb{N}$, $k$ not a power of two, there are cyclic vectors of length $k$ which are not the concatenation of two or more copies of a vector of smaller length. As an application of this, we give a new proof of the fact that the period of a Ducci sequence can be any positive integer with the exception of the powers of 2 greater than 1.

1. INTRODUCTION TO DUCCI SEQUENCES

Let $k \in \mathbb{N}$ and let $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$. We define a map $T: \mathbb{N}^k \to \mathbb{N}^k$ by

$$T(\vec{x}) = T(a_0, a_1, \ldots, a_{k-1}) = (|a_0 - a_1|, |a_1 - a_2|, \ldots, |a_{k-1} - a_0|).$$

The sequence $(T^n(\vec{x}))_{n \in \mathbb{N}}$ generated by the iterations of $T$ is called a Ducci sequence. Ducci sequences have been extensively studied and often rediscovered. We state a few well-known facts which will be used in this paper.

Let $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$. If there exists $a \in \mathbb{N}$ such that for every $0 \leq i \leq k$, $a_i \in \{0, a\}$, we will say that $\vec{x}$ is a simple vector. A well-known result states that for every $\vec{x} \in \mathbb{N}^k$, there exists $n \in \mathbb{N}$ such that $T^n(\vec{x})$ is simple (see [5] for example). Since there are only finitely many vectors of length $k$ with components in $\{0, a\}$, the iterations must eventually repeat. In other words, every Ducci sequence is ultimately cyclic. We derive another important consequence of this fact.

Let $\vec{x} = (a_0, a_1, \ldots, a_{k-1})$ be a simple vector with $a_i \in \{0, a\}$ for every $0 \leq i \leq k-1$. We can rewrite $\vec{x}$ as $(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1})$, where $\epsilon_i = 1$ if $a_i = a$ and 0 otherwise. Notice that for every $k \in \mathbb{N}$, $T^k(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1}) = a \cdot T^k(\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$. This simple remark implies that in order to study cycles in Ducci sequences, we can restrict our attention to vectors with components in $\{0, 1\}$.

Finally, note that when $a_i \in \{0, 1\}$, the operation $|a_i - a_{i+1}|$ is equivalent to $a_i + a_{i+1}$ (mod 2). Consequently, we will study the map $T: \mathbb{Z}_2^k \to \mathbb{Z}_2^k$ defined by

$$T(a_0, a_1, \ldots, a_{k-1}) = (a_0 + a_1, a_1 + a_2, \ldots, a_{k-1} + a_0).$$

We will call the sequences generated by iterating this map Ducci sequences over $\mathbb{Z}_2$. If for some vector $\vec{x} \in \mathbb{N}^k$, there exists an integer $p$ such that $T^p(\vec{x}) = \vec{x}$, and $p$ is minimal with this property, we will say that $\vec{x}$ has period $p$. The discussion above implies that in order to study the periods of the Ducci sequences it is sufficient to do so over $\mathbb{Z}_2$. We will do so in the remainder of this paper.

2. PROOF OF THE MAIN RESULT

In order to simplify the notation, the indices of the components of any vector $\vec{x} \in \mathbb{Z}_2^k$ will be written modulo $k$ so that, for example, $a_k = a_0$ and $a_{k+1} = a_1$.

If $T^m(\vec{x}) = \vec{x}$ for some integer $m$, we will say that $\vec{x}$ is cyclic. If $m$ is the smallest such integer, we say that $\vec{x}$ is $m$-cyclic, or as defined earlier, has period $m$. If for some integer $n$,
we have $T^n(\vec{x}) = \vec{y}$ and $\vec{y}$ is $m$-cyclic, we say that $\vec{y}$ belongs to the cycle generated by $\vec{x}$ and we write $c(\vec{x}) = m$. In other words, $c(\vec{x})$ is the length of the cycle that the iterations of $\vec{x}$ will eventually reach. If $T^n(\vec{x}) = \vec{0}$ for some integer $m$, we say that $\vec{x}$ is nilpotent.

Given $k, l \in \mathbb{N}$ and two vectors $\vec{x} = (x_0, x_1, \ldots, x_k)$ and $\vec{y} = (y_0, y_1, \ldots, y_l)$ we denote by $\vec{x} \vee \vec{y}$ their concatenation $(x_0, x_1, \ldots, x_k, y_1, y_2, \ldots, y_l)$. For $m > 1$, we write $\vec{x} \vee \vec{x} \vee \cdots \vee \vec{x} = \vee^{(m)} \vec{x}$, the concatenation of $\vec{x}$ with itself $m$ times and by convention $\vee^{(1)} \vec{x} = \vec{x}$.

It is easy to see that for any $\vec{x} \in \mathbb{Z}_2^k$ (or in $\mathbb{N}^k$) and for any positive integers $n, m$ the following relation holds:

$$T^n(\vee^{(m)} \vec{x}) = \vee^{(m)} T^n(\vec{x}). \quad (2.1)$$

In particular, $T^n(\vec{x}) = \vec{0}$ implies $T^n(\vee^{(m)} \vec{x}) = \vec{0}$ for every $m$.

A fundamental theorem states that if $k = 2^l$ for some $l$, then any $\vec{x} \in \mathbb{Z}_2^k$ is nilpotent (see for example [3]). Together with (2.1), this implies that for any $m \in \mathbb{N}$ and any $\vec{x} \in \mathbb{Z}_2^k$, the vector $\vee^{(m)} \vec{x}$ is nilpotent. In [1] a converse to this statement was proved, thus showing the following proposition.

**Proposition 2.1.** Let $\vec{x} \in \mathbb{Z}_2^k$. The vector $\vec{x}$ is nilpotent if and only if there exist $l, m \in \mathbb{N}$ and $\vec{y} \in \mathbb{Z}_2^l$ such that $\vec{x} = \vee^{(m)} \vec{y}$.

The equality (2.1) also allows us to construct new cyclic vectors: if $\vec{x}$ is cyclic of period $p$, so is $\vee^{(m)} \vec{x}$ for any $m > 1$. The main result of this paper is that for any $k \in \mathbb{N}$, $k$ not a power of two, there are cyclic vectors in $\mathbb{Z}_2^k$ (and thus also in $\mathbb{N}^k$) that are not the concatenation of two or more copies of a smaller vector. So in essence, for every $k$ which is not a power of 2, $\mathbb{Z}_2^k$ has “original” cyclic vectors. Note that this result is obvious if $k > 2$ is prime and not true if $k$ is a power of 2 since in this case only $\vec{0}$ is cyclic. We will need two simple lemmas.

**Lemma 2.2.** Let $k \in \mathbb{N}$ and $\vec{x}$ be any element of $\mathbb{Z}_2^k$. There exists a unique $\vec{y}$ in the cycle generated by $\vec{x}$ and a unique nilpotent $\vec{z}$ such that $\vec{x} = \vec{y} + \vec{z}$.

**Proof.** First we show the existence. If $\vec{x}$ belongs to the cycle generated by itself, take $\vec{z} = \vec{0}$. Otherwise, choose $n$ such that $\vec{c} = T^n(\vec{x})$ is cyclic. Denote by $m$ the period of $\vec{c}$ and choose $l$ such that $lm > n$. Define $\vec{z} = \vec{x} + T^{lm-n}(\vec{c})$. The linearity of $T$ over $\mathbb{Z}_2^k$ implies that $T^n(\vec{z}) = T^n(\vec{x}) + T^{n+lm-n}(\vec{c}) = \vec{c} + T^{lm}(\vec{c}) = \vec{c} + \vec{c} = \vec{0}$. In other words, $\vec{z}$ is nilpotent and we can write $\vec{x} = T^{lm-n}(\vec{c}) + \vec{z}$.

To show uniqueness, suppose $\vec{x} = \vec{y} + \vec{z} = \vec{y}' + \vec{z}'$ where $\vec{y}$ and $\vec{y}'$ both belong to the cycle generated by $\vec{x}$ and $\vec{z}, \vec{z}'$ are both nilpotent. Let $n$ be an integer such that $T^n(\vec{z}) = T^n(\vec{z}') = \vec{0}$. Then

$$T^n(\vec{y} + \vec{z}) = T^n(\vec{y}' + \vec{z}')$$

$$\Rightarrow T^n(\vec{y}) + T^n(\vec{z}) = T^n(\vec{y}') + T^n(\vec{z}')$$

$$\Rightarrow T^n(\vec{y}) = T^n(\vec{y}')$$

But since both $\vec{y}$ and $\vec{y}'$ belong to the same cycle we must have $\vec{y} = \vec{y}'$. It follows that $\vec{z} = \vec{z}'$.

Define $\text{Nil}_k$ to be the set of nilpotent vectors in $\mathbb{Z}_2^k$. Lemma 2.2 immediately implies the following.

**Lemma 2.3.** Let $\vec{x} \in \mathbb{Z}_2^k$. Exactly one element in the set $\{\vec{x} + \vec{z}, \vec{z} \in \text{Nil}_k\}$ is cyclic.
Proof. Suppose that there are two different \( \vec{z}_1 \) and \( \vec{z}_2 \) in \( \text{Nil}_k \) such that \( \vec{x} + \vec{z}_1 = \vec{y}_1 \) and \( \vec{x} + \vec{z}_2 = \vec{y}_2 \) are both cyclic. Then we can rewrite these equalities as \( \vec{x} = \vec{y}_1 + \vec{z}_1 \) and \( \vec{x} = \vec{y}_2 + \vec{z}_2 \), contradicting the previous lemma.

Define \( C_k \) to be the set of cyclic elements of \( \mathbb{Z}_2^k \) and
\[
S_k = \{ \vec{x} \in C_k : \text{there exists } \vec{y} \text{ such that } \vec{x} = \vee(n)\vec{y} \text{ for some } n \geq 2 \}.
\]

**Theorem 2.4.** For every positive integer \( k \), \( k \) not a power of 2, there is at least one cyclic vector of length \( k \), which is not the concatenation of two or more copies of a smaller vector.

Proof. Set \( k = 2^l m \) for some odd number \( m \geq 3 \). The statement is equivalent to showing that the set \( C_k \setminus S_k \) is non-empty. There are exactly \( 2^m \) vectors in \( \mathbb{Z}_2^k \) that can be obtained by concatenating vectors of length \( m \) a power of 2. Proposition 2.1 implies \( |\text{Nil}_k| = 2^l \). Using Lemma 2.3 we obtain \( |C_k||\text{Nil}_k| = |\mathbb{Z}_2^k| \) or \( |C_k|2^l = 2^m \). Consequently,
\[
|C_k| = 2^{l(m-1)}. \quad (2.2)
\]

Notice that if \( \vec{x} = \vee(n)\vec{y} \) for \( n \geq 2 \) then \( \vec{y} \) can be of length at most \( k/2 \). Thus, the values of the first \( \lfloor k/2 \rfloor \) components of \( \vec{x} \) determine \( \vec{x} \) entirely. Since the vector \( (1, 1, \ldots, 1) \) is not in \( C_k \), we have the following upper bound on the size of \( S_k \):
\[
|S_k| \leq 2^{\lfloor k/2 \rfloor} - 1 \leq 2^{2^{l-1}m} - 1.
\]

The last inequality in conjunction with (2.2) gives us the following lower bound on the size of \( C_k \setminus S_k \)
\[
|C_k - S_k| \geq 2^{l(m-1)} - (2^{2^{l-1}m} - 1) \geq 1, \quad (2.3)
\]
since \( m \geq 3 \), concluding the proof.

The equality (2.3) in the above proof was first proved by Ludington-Young in [7] and later in [2] by Brown and Merzel. If \( 2_k \) denotes the highest power of 2 dividing \( k \), then it can be restated as the following corollary.

**Corollary 2.5** (Young’s Theorem). For every \( k \in \mathbb{N} \), the number of cyclic vectors is \( 2^{k-2_k} \).

If we denote by \( R \) the rotation of components defined by \( R((a_0, a_1, \ldots, a_{k-1})) = (a_1, a_2, \ldots, a_0) \), then \( R(\vec{x}) \) is cyclic whenever \( \vec{x} \) is cyclic. Note also that if \( \vec{x} \) is not the concatenation of two or more copies of a shorter vector, then for any \( n \in \mathbb{N} \), \( R^n(\vec{x}) \neq \vec{x} \). Consequently, we actually have \( |C_k \setminus S_k| \geq k \), strengthening Theorem 2.4.

**Corollary 2.6.** For any \( k \) not a power of 2, \( |C_k \setminus S_k| \geq k \).

### 3. An Application of Theorem 2.4

In this section, we begin by proving that the period of a Ducci sequence cannot be a power of 2 greater than 1 (the null vector has period 1).

**Proposition 3.1.** For every integer \( m \geq 1 \) and every \( \vec{x} \in \mathbb{Z}_2^k \), \( c(\vec{x}) \neq 2^m \).

Proof. Suppose \( \vec{x} \) is a cyclic vector such that \( c(\vec{x}) = 2^m \) for some positive integer \( m \). Consider the vector \( \vec{x}_1 = \vec{x} + T^{2^m-1}(\vec{x}) \). It cannot be that \( \vec{x}_1 = \vec{0} \), otherwise \( \vec{x} = T^{2^m-1}(\vec{x}) \) and \( c(\vec{x}) \leq 2^{m-1} \), contradicting our assumption. Also notice that
\[
T^{2^m-1}(\vec{x}_1) = T^{2^m-1}(\vec{x} + T^{2^m-1}(\vec{x})) = T^{2^m-1}(\vec{x}) + T^{2^m}(\vec{x}) = T^{2^m-1}(\vec{x}) + \vec{x} = \vec{x}_1.
\]
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Thus from $\vec{x}$ we constructed a non-zero cyclic vector $\vec{x}_1$ whose period $c(\vec{x}_1)$ must be a divisor of $2^{m-1}$. If we repeat the procedure with $\vec{x}_1$, we obtain a cyclic vector $\vec{x}_2$ whose period divides $2^{m-2}$ and $\vec{x}_2 \neq \vec{0}$. By repeating the process $m$ times, we create a non-null cyclic vector $\vec{x}_m$ with $c(\vec{x}_m) = 1$, a contradiction. \hfill $\Box$

We use Theorem 2.4 to show that any positive integer which is a power of 2 is the period of a Ducci sequence. This fact is also the consequence of a more general result proved in [2]. Related results were also proved earlier in [7]. Both [7, 2] exploit a form of duality between the “rows” and “columns” of Ducci sequences. We use the same idea in the following proof.

Theorem 3.2. Let $m$ be an integer. If $m \neq 2^t$ for any $t \geq 2$, then there exist $k$ and a vector $\vec{x}$ in $\mathbb{Z}_k$ such that $c(\vec{x}) = m$.

Proof. Let $k = 2^l \cdot q$ for some integers $l, q$ where $q \geq 3$ is odd. Using Theorem 2.4, let $\vec{x} = \{x_0, x_1, \ldots, x_{k-1}\} \in C_k \setminus S_k$ and form the matrix $M = [a_{i,j}], 1 \leq i \leq c(\vec{x}), 0 \leq j \leq k-1$ where $a_{i,j}$ is the $j$ component of $T^i(\vec{x})$. By construction

$$a_{i,j} + a_{i,j+1} = a_{i+1,j}$$

or equivalently

$$a_{i+1,j} = a_{i,j+1}. \quad (3.1)$$

Consider for $0 \leq i \leq k-1$ the vectors $\vec{C}_i = (a_{c(\vec{x})-i}, a_{c(\vec{x})-1-i}, \ldots, a_{1-i}),$ the transposition of the $i + 1$ column of $M$. Note that the role of the indices is now reversed. In particular $i$ now represent the row. By (3.1) we have $T(\vec{C}_i) = \vec{C}_{i+1}$, where as usual the addition of the index is taken modulo $k$. In particular

$$T^k(\vec{C}_1) = \vec{C}_1. \quad (3.2)$$

We claim that $c(\vec{C}_1) = k$. By (3.2) we have $c(\vec{C}_1) | k$. Suppose that $c(\vec{C}_1) = t$ is a proper divisor of $k$. Then $T^t(\vec{C}_1) = \vec{C}_1$ implying $x_0 = x_t$. Similarly, since then $T^t(C_i) = C_i$ for every $0 \leq i \leq k-1$ we obtain in general

$$x_i = x_{i+t}.$$ But then $\vec{x} = (x_0, x_1, \ldots, x_{t-1}, x_0, \ldots, x_{t-1}, \ldots, x_0, \ldots, x_{t-1}),$ contradicting the fact that $\vec{x} \notin S_k$. \hfill $\Box$

References

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Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia 30303
E-mail address: cavart@gsu.edu