THE ASYMPTOTIC BEHAVIOR OF $\prod_{k=0}^{n}\binom{n}{k}$

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Abstract. We determine the complete asymptotic expansion of $\prod_{k=0}^{n}\binom{n}{k}$.

1. Introduction

Harlan J. Brothers [2] raised the problem of determining the asymptotic behavior of $\prod_{k=0}^{n}\binom{n}{k}$.

In this note we prove the following.

Theorem.

$$\prod_{k=0}^{n}\binom{n}{k} \sim C^{-1}\frac{e^{n(n+2)/2}}{n^{(3n+2)/6}(2\pi)^{(2n+1)/4}}\exp\left\{-\sum_{p\geq 1}\frac{B_{p+1}+B_{p+2}}{p(p+1)}\frac{1}{n^p}\right\} \quad \textit{as } n\to\infty,$$

where

$$C = \lim_{n\to\infty}\frac{1}{n^{\frac{1}{12}}}\prod_{k=1}^{n}\left\{k!\Big/\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right\}$$
$$\approx 1.04633506677050318098095065697776 0...$$

and the $\{B_p\}$ *are the Bernoulli numbers, defined by*

$$\sum_{p\geq 0}B_p\frac{x^p}{p!} = \frac{x}{e^x-1}.$$

2. Preliminaries

We shall require the following preliminary results.

Lemma 1. *(Stirling's Formula)*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\exp\left\{\sum_{p\geq 1}\frac{B_{p+1}}{p(p+1)}\frac{1}{n^p}\right\} \quad \textit{as } n\to\infty.$$

Proof. See [1], Theorem 1.4.2. □

Lemma 2.

$$\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}\Big/e\right\}=\frac{e}{\sqrt{2\pi}}.$$

Proof.

$$\begin{aligned}\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}\Big/e\right\}&=\prod_{k=1}^{\infty}\left\{\frac{k!}{\sqrt{k}\left(\frac{k}{e}\right)^{k}}\Big/\frac{(k+1)!}{\sqrt{k+1}\left(\frac{k+1}{e}\right)^{k+1}}\right\}\\&=e\Big/\lim_{k\to\infty}\frac{(k+1)!}{\sqrt{k+1}\left(\frac{k+1}{e}\right)^{k+1}}\\&=e/\sqrt{2\pi}\end{aligned}$$

□

by Lemma 1.

Lemma 3.

$$\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}=\frac{e}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\left\{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/k!.\right\}$$

Proof.

$$\begin{aligned}&\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}\\&=\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\prod_{l=1}^{k}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\Big/\prod_{l=1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}\\&=\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\frac{\sqrt{2\pi}}{e}\prod_{l=1}^{k}\left\{\frac{l!}{\sqrt{l}\left(\frac{l}{e}\right)^{l}}\Big/\frac{(l+1)!}{\sqrt{l+1}\left(\frac{l+1}{e}\right)^{l+1}}\right\}\right\}\\&=\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\frac{\sqrt{2\pi}}{e}\left\{e\Big/\frac{(k+1)!}{\sqrt{k+1}\left(\frac{k+1}{e}\right)^{k+1}}\right\}\right\}\\&=\prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\frac{\sqrt{2\pi(k+1)}\left(\frac{k+1}{e}\right)^{k+1}}{(k+1)!}\right\}\\&=\prod_{k=1}^{\infty}\left\{\left\{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/k!\right\}\left\{\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}\Big/e\right\}\right\}\\&=\frac{e}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\left\{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/k!\right\}.\end{aligned}$$

□

Lemma 4. *For $p\geq 2$,*

$$\sum_{k=n+1}^{\infty}\frac{1}{k^{p}}\sim\sum_{k\geq 0}\frac{(p-2+k)!B_{k}}{(p-1)!k!}\frac{1}{n^{p-1+k}}.$$

Proof. See [3] p. 502, B.24. □

Lemma 5.

$$\sum_{k=n}^{\infty} \frac{1}{k^p} \sim \sum_{k\geq 0} \frac{(p-2+k)!(-1)^k B_k}{(p-1)!k!} \frac{1}{n^{p-1+k}}.$$

Proof. Add $\frac{1}{n^p}$ to the result in Lemma 4. (Only the term $k=1$ is affected.) □

Lemma 6.

$$\sum_{p\geq 2} (-1)^p \frac{p-1}{2p(p+1)} \sum_{k\geq 0} \frac{(p-2+k)!B_k}{(p-1)!k!} \frac{1}{n^{p-1+k}}$$
$$= \sum_{p\geq 1} (-1)^{p+1} \left(\frac{p-1}{2p(p+1)} + \frac{B_{p+1}}{p(p+1)} \right) \frac{1}{n^p}.$$

Proof. With $p-1+k=P$, we have

$$\sum_{p\geq 2,\ k\geq 0} (-1)^p \frac{p-1}{2p(p+1)} \frac{(p-2+k)!B_k}{(p-1)!k!} \frac{1}{n^{p-1+k}}$$
$$= \sum_{p\geq 2,\ k\geq 0} (-1)^p \frac{(p-1)(p-2+k)!B_k}{2(p+1)!k!} \frac{1}{n^{p-1+k}}$$
$$= \sum_{P\geq 1} \frac{1}{n^P} \sum_{k=0}^{P-1} \frac{(-1)^{P-k+1}(P-k)(P-1)!B_k}{2(P-k+2)!k!}$$
$$= \sum_{P\geq 1} \frac{(-1)^{P+1}}{n^P} \sum_{k=0}^{P-1} \frac{(-1)^k \left((P-k+2)-2\right)(P-1)!B_k}{2(P-k+2)!k!}$$
$$= \sum_{P\geq 1} \frac{(-1)^{P+1}}{n^P} \left(\sum_{k=0}^{P-1} \frac{(-1)^k (P-1)!B_k}{2(P-k+1)!k!} - \sum_{k=0}^{P-1} \frac{(-1)^k (P-1)!B_k}{(P-k+2)!k!} \right)$$

The internal sum is

$$\frac{(P-1)!}{2} \sum_{k=0}^{P-1} [x^k] \left(\frac{xe^x}{e^x-1} \right) [x^{P-1-k}] \left(\frac{e^x-1-x}{x^2} - 2\frac{e^x-1-x-x^2/2}{x^3} \right)$$
$$= \frac{(P-1)!}{2} [x^{P-1}] \left(\left(\frac{xe^x}{e^x-1} \right) \left(\frac{e^x-1-x}{x^2} - 2\frac{e^x-1-x-x^2/2}{x^3} \right) \right)$$
$$= \frac{(P-1)!}{2} [x^{P-1}] \left(\frac{2}{x} + \frac{e^x}{x} - \frac{2e^x}{x^2} + \frac{2}{x^2} \left(\frac{x}{e^x-1} \right) \right)$$
$$= \frac{(P-1)!}{2} [x^{P-1}] \left(\frac{2}{x} + \sum_{k\geq 0} \frac{x^{k-1}}{k!} - 2\sum_{k\geq -1} \frac{x^{k-1}}{(k+1)!} + 2\sum_{k\geq -1} \frac{2B_{k+1}x^{k-1}}{(k+1)!} \right)$$
$$= \frac{(P-1)!}{2} \left(\frac{1}{P!} - 2\frac{1}{(P+1)!} + \frac{2B_{P+1}}{(P+1)!} \right)$$
$$= \frac{P-1}{2P(P+1)} + \frac{B_{P+1}}{P(P+1)}.$$

Thus the sum becomes

$$\sum_{P\geq 1}(-1)^{P+1}\left(\frac{P-1}{2P(P+1)}+\frac{B_{P+1}}{P(P+1)}\right)\frac{1}{n^P}.$$

□

Lemma 7.

$$\sum_{p\geq 2}(-1)^p\left(\frac{p-1}{2p(p+1)}-\frac{1}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!(-1)^k B_k}{(p-1)!k!}\frac{1}{n^{p-1+k}}$$
$$=\sum_{p\geq 1}\left(\frac{B_{p+1}}{2p(p+1)}+\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}.$$

Proof. With $P=p-1+k$ as before, we have

$$\sum_{p\geq 2}(-1)^p\left(\frac{p-1}{2p(p+1)}-\frac{1}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!(-1)^k B_k}{(p-1)!k!}\frac{1}{n^{p-1+k}}$$
$$=\sum_{p\geq 2,\ k\geq 0}\frac{(-1)^{p+k}}{n^{p-1+k}}\left(\frac{p-1}{2p(p+1)}-\frac{1}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\frac{(p-2+k)!B_k}{(p-1)!k!}$$
$$=\sum_{P\geq 1}\frac{(-1)^{P+1}}{n^P}\sum_{k=0}^{P-1}\left(\frac{(P-k)(P-1)!B_k}{2(P-k+2)!k!}-\frac{(P-1)!B_k}{12(P-k+1)!k!}+\frac{(P-1)!}{(P-k+2)!k!}B_kB_{P+2-k}\right)$$

The first internal sum is

$$\sum_{k=0}^{P-1}\frac{(P-k)(P-1)!B_k}{2(P-k+2)!k!}$$
$$=\sum_{k=0}^{P-1}\frac{((P-k+2)-2)\,(P-1)!B_k}{2(P-k+2)!k!}$$
$$=\sum_{k=0}^{P-1}\frac{(P-1)!B_k}{2(P-k+1)!k!}-\sum_{k=0}^{P-1}\frac{(P-1)!B_k}{(P-k+2)!k!}$$
$$=\frac{(P-1)!}{2}\sum_{k=0}^{P-1}[x^k]\left(\frac{x}{e^x-1}\right)[x^{P-1-k}]\left(\frac{e^x-1-x}{x^2}-2\frac{e^x-1-x-x^2/2}{x^3}\right)$$
$$=\frac{(P-1)!}{2}[x^{P-1}]\left(\left(\frac{x}{e^x-1}\right)\left(\frac{e^x-1-x}{x^2}-2\frac{e^x-1-x-x^2/2}{x^3}\right)\right)$$
$$=\frac{(P-1)!}{2}[x^{P-1}]\left(\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^2}\left(\frac{x}{e^x-1}\right)\right)$$
$$=\frac{(P-1)!}{2}[x^{P-1}]\left(\frac{1}{x}-\frac{2}{x^2}+2\sum_{k\geq -1}\frac{B_{k+1}x^{k-1}}{(k+1)!}\right)$$
$$=\frac{B_{P+1}}{P(P+1)}.$$

The second internal sum is

$$\begin{aligned}
-\sum_{k=0}^{P-1}\frac{(P-1)!B_k}{12(P-k+1)!k!} &= -\frac{(P-1)!}{12}\sum_{k=0}^{P-1}[x^k]\left(\frac{x}{e^x-1}\right)[x^{P-1-k}]\left(\frac{e^x-1-x}{x^2}\right)\\
&= -\frac{(P-1)!}{12}[x^{P-1}]\left(\left(\frac{x}{e^x-1}\right)\left(\frac{e^x-1-x}{x^2}\right)\right)\\
&= -\frac{(P-1)!}{12}[x^{P-1}]\left(\frac{1}{x}-\frac{1}{x}\left(\frac{x}{e^x-1}\right)\right)\\
&= -\frac{(P-1)!}{12}[x^{P-1}]\left(\frac{1}{x}-\sum_{k\geq 0}\frac{B_p x^{P-1}}{P!}\right)\\
&= \frac{B_P}{12P}.
\end{aligned}$$

The third internal sum is

$$\begin{aligned}
&\sum_{k=0}^{P-1}\frac{(P-1)!}{(P-k+2)!k!}B_kB_{P+2-k}\\
&= (P-1)!\sum_{k=0}^{P+2}[x^k]\left(\frac{x}{e^x-1}\right)[x^{P+2-k}]\left(\frac{x}{e^x-1}\right)-\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}\\
&= (P-1)![x^{P+2}]\left(\left(\frac{x}{e^x-1}\right)^2\right)-\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}\\
&= (P-1)![x^{P+2}]\left(B(x)^2\right)-\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}.
\end{aligned}$$

Now,

$$xB'(x) = B(x) - xB(x) - B(x)^2$$

so

$$B(x)^2 = B(x) - xB(x) - xB'(x).$$

So the third internal sum is

$$\begin{aligned}
&(P-1)![x^{P+2}]\left(B(x)-xB(x)-xB'(x)\right)-\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}\\
&= (P-1)![x^{P+2}]\left(\sum_{k\geq -2}\frac{B_{k+2}x^{k+2}}{(k+2)!}-\sum_{k\geq -1}\frac{B_{k+1}x^{k+2}}{(k+1)!}-\sum_{k\geq -1}\frac{B_{k+2}x^{k+2}}{(k+1)!}\right)\\
&\quad -\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}\\
&= \frac{B_{P+2}}{P(P+1)P+2)}-\frac{B_{P+1}}{P(P+1)}-\frac{B_{P+2}}{P(P+1)}-\frac{B_P}{12P}+\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)(P+2)}\\
&= -\frac{B_P}{12P}-\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)}.
\end{aligned}$$

So the sum becomes

$$\sum_{P\geq 1}(-1)^{P+1}\left(\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)}\right)\frac{1}{n^P}=\sum_{P\geq 1}\left(\frac{B_{P+1}}{2P(P+1)}+\frac{B_{p+2}}{P(P+1)}\right)\frac{1}{n^P}.$$

□

Lemma 8.

$$\sum_{p\geq 2}\left(\frac{(-1)^p}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!B_k}{(p-1)!k!}\frac{1}{n^{p-1+k}}$$
$$=\sum_{p\geq 1}\left(\frac{(-1)^{p+1}}{12p}-\frac{B_{p+1}}{2p(p+1)}-\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}.$$

Proof. Consider the sum

$$\sum_{p\geq 2}\left(\frac{(-1)^p}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!B_k}{(p-1)!k!}\frac{1}{n^{p-1+k}}.$$

We split this in two. The first sum is

$$\frac{1}{12}\sum_{p\geq 2,\ k\geq 0}\frac{(-1)^p(p-2+k)!B_k}{p!k!}\frac{1}{n^{p-1+k}}$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{1}{n^P}\sum_{k=0}^{P-1}\frac{(-1)^{P+1+k}(P-1)!B_k}{(P+1-k)!k!}$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{(-1)^{P+1}}{n^P}(P-1)!\sum_{k=0}^{P-1}\frac{(-1)^kB_k}{(P+1-k)!k!}$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{(-1)^{P+1}}{n^P}(P-1)!\left(\sum_{k=0}^{P+1}\frac{(-1)^kB_k}{(P+1-k)!k!}-\frac{(-1)^PB_P}{P!}-\frac{(-1)^{P+1}B_{P+1}}{(P+1)!}\right)$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{1}{n^P}\left((-1)^{P+1}(P-1)!\sum_{k=0}^{P+1}\frac{(-1)^kB_k}{(P+1-k)!k!}+\frac{B_P}{P}-\frac{B_{P+1}}{P(P+1)}\right)$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{1}{n^P}\left((-1)^{P+1}(P-1)![x^{P+1}]\left(e^xB(-x)\right)+\frac{B_P}{P}-\frac{B_{P+1}}{P(P+1)}\right)$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{1}{n^P}\left((-1)^{P+1}(P-1)![x^{P+1}]\left(x+xe^x+\frac{x}{e^x-1}\right)+\frac{B_P}{P}-\frac{B_{P+1}}{P(P+1)}\right)$$
$$=\frac{1}{12}\sum_{P\geq 1}\frac{1}{n^P}\left((-1)^{P+1}(P-1)!\left(\frac{1}{P!}+\frac{B_{P+1}}{(P+1)!}\right)+\frac{B_P}{P}-\frac{B_{P+1}}{P(P+1)}\right)$$
$$=\sum_{P\geq 1}\frac{1}{n^P}\left(\frac{(-1)^{P+1}}{12P}+\frac{(-1)^{P+1}B_{P+1}}{12P(P+1)}+\frac{B_P}{12P}-\frac{B_{P+1}}{12P(P+1)}\right)$$
$$=\sum_{P\geq 1}\frac{1}{n^P}\left(\frac{(-1)^{P+1}}{12P}+\frac{B_P}{12P}\right).$$

The second sum is

$$\begin{aligned}
&\sum_{p\geq 2,\ k\geq 0}\frac{B_{p+1}}{p(p+1)}\frac{(p-2+k)!B_k}{(p-1)!k!}\frac{1}{n^{p-1+k}}\\
&=\sum_{P\geq 1}\frac{1}{n^P}\sum_{k=0}^{P-1}\frac{(P-1)!B_{P+2-k}B_k}{(P-k+2)!k!}\\
&=\sum_{P\geq 1}\frac{1}{n^P}\left(-\frac{B_P}{12P}-\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)}\right).
\end{aligned}$$

(This calculation was done in the proof of Lemma 7.)
Combining the two, we obtain the sum

$$\sum_{P\geq 1}\left(\frac{(-1)^{P+1}}{12P}-\frac{B_{P+1}}{2P(P+1)}-\frac{B_{P+2}}{P(P+1)}\right)\frac{1}{n^P}.$$

□

3. The Derivation

Let

$$P_n=\prod_{k=0}^{n}\binom{n}{k}=\frac{(n!)^{n+1}}{0!^2 1!^2\cdots n!^2}$$

and

$$r_n=\frac{P_{n+1}}{P_n}=\frac{(n+1)!^{n+2}}{n!^{n+1}(n+1)!^2}=\frac{(n+1)!^n}{n!^{n+1}}=\frac{(n+1)^n}{n!}.$$

Then

$$\begin{aligned}
\frac{r_n}{r_{n-1}}&=\frac{(n+1)^n}{n^{n-1}}\cdot\frac{(n-1)!}{n!}=\left(1+\frac{1}{n}\right)^n\\
&=\exp\left\{n\log\left(1+\frac{1}{n}\right)\right\}\\
&=\exp\left\{n\left(\sum_{p\geq 1}\frac{(-1)^{p+1}}{pn^p}\right)\right\}\\
&=\exp\left\{1+\sum_{p\geq 1}\frac{(-1)^p}{(p+1)n^p}\right\}\\
&=e\cdot\exp\left\{-\frac{1}{2}\sum_{p\geq 1}\frac{(-1)^{p+1}}{pn^p}+\sum_{p\geq 2}(-1)^p\left(\frac{1}{p+1}-\frac{1}{2p}\right)\frac{1}{n^p}\right\}\\
&=e\cdot\exp\left\{-\frac{1}{2}\log\left(1+\frac{1}{n}\right)+\sum_{p\geq 2}(-1)^p\frac{p-1}{2p(p+1)}\frac{1}{n^p}\right\}
\end{aligned}$$

$$= e \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \exp\left\{ \sum_{p\geq 2} (-1)^p \frac{p-1}{2p(p+1)} \frac{1}{n^p} \right\}. \tag{3.1}$$

In fact, (3.1) is nothing more than

$$\frac{r_n}{r_{n-1}} = e\frac{\sqrt{n}}{\sqrt{n+1}} \exp\left\{ \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) - 1 \right\}$$

$$= e\frac{\sqrt{n}}{\sqrt{n+1}} \left\{ \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \Big/ e \right\}. \tag{3.2}$$

It follows from (3.2) that, since $r_0 = 1$,

$$r_n = \frac{r_n}{r_{n-1}} \cdot \frac{r_{n-1}}{r_{n-2}} \cdot \ \cdots \ \frac{r_1}{r_0} r_0$$

$$= \frac{e^n}{\sqrt{n+1}} \prod_{k=1}^{n} \left\{ \left(1+\frac{1}{k}\right)^{k+\frac{1}{2}} \Big/ e \right\}. \tag{3.3}$$

It follows from (3.3) and Lemma 2 that

$$r_n \Big/ \left\{ \frac{e^n}{\sqrt{n+1}} \right\} \to \prod_{k=1}^{\infty} \left\{ \left(1+\frac{1}{k}\right)^{k+\frac{1}{2}} /e \right\} = \frac{e}{\sqrt{2\pi}}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$r_n = \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \Big/ \prod_{k=n+1}^{\infty} \left\{ \left(1+\frac{1}{k}\right)^{k+\frac{1}{2}} \Big/ e \right\}. \tag{3.5}$$

From (3.5) and (3.1) we have

$$r_n = \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \exp\left\{ -\sum_{p\geq 2} (-1)^p \frac{p-1}{2p(p+1)} \sum_{k=n+1}^{\infty} \frac{1}{k^p} \right\}. \tag{3.6}$$

From (3.6) and Lemma 4 we have

$$r_n \sim \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \exp\left\{ -\sum_{p\geq 2} (-1)^p \frac{p-1}{2p(p+1)} \sum_{k\geq 0} \frac{(p-2+k)!B_k}{(p-1)!k!} \frac{1}{n^{p-1+k}} \right\}. \tag{3.7}$$

From (3.7) and Lemma 6 we have

$$r_n \sim \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \exp\left\{ \sum_{p\geq 1} (-1)^p \left(\frac{p-1}{2p(p+1)} + \frac{B_{p+1}}{p(p+1)} \right) \frac{1}{n^p} \right\}. \tag{3.8}$$

We can write (3.8)

$$r_n \sim \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{12}\log\left(1+\frac{1}{n}\right) + \sum_{p\geq 2} (-1)^p \left(\frac{p-1}{2p(p+1)} - \frac{1}{12p} + \frac{B_{p+1}}{p(p+1)} \right) \frac{1}{n^p} \right\}$$

$$\sim \frac{e^n}{\sqrt{n+1}} \frac{e}{\sqrt{2\pi}} \left(\frac{n}{n+1}\right)^{\frac{1}{12}} \exp\left\{ \sum_{p\geq 2} (-1)^p \left(\frac{p-1}{2p(p+1)} - \frac{1}{12p} + \frac{B_{p+1}}{p(p+1)} \right) \frac{1}{n^p} \right\}. \tag{3.9}$$

We see from (3.5) that (3.9) is nothing other than

$$r_n = \frac{e^n}{\sqrt{n+1}}\frac{e}{\sqrt{2\pi}}\left(\frac{n}{n+1}\right)^{\frac{1}{12}}\left\{\left(1+\frac{1}{n}\right)^{\frac{1}{12}}\Big/\prod_{k=n+1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{k+\frac{1}{2}}\Big/e\right\}\right\}. \tag{3.10}$$

From (3.10) we obtain, since $P_1 = 1$,

$$\begin{aligned}P_n &= r_{n-1}\cdot r_{n-2}\cdot\ \cdots\ \cdot r_1 P_1\\ &= \frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\cdot\prod_{k=1}^{n-1}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}.\end{aligned} \tag{3.11}$$

We can write (3.11)

$$P_n\Big/\left\{\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\right\} = \prod_{k=1}^{n-1}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}. \tag{3.12}$$

It follows from (3.12) that as $n\to\infty$,

$$P_n\Big/\left\{\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\right\} \to A = \prod_{k=1}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}. \tag{3.13}$$

By Lemma 3,

$$A = \frac{e}{\sqrt{2\pi}}\Big/\prod_{k=1}^{\infty}\left\{k!\Big/\left\{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\right\}\right\}. \tag{3.14}$$

It follows from (3.12) and (3.13) that

$$P_n = A\,\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\Big/\prod_{k=n}^{\infty}\left\{\left(1+\frac{1}{k}\right)^{\frac{1}{12}}\Big/\prod_{l=k+1}^{\infty}\left\{\left(1+\frac{1}{l}\right)^{l+\frac{1}{2}}\Big/e\right\}\right\}. \tag{3.15}$$

From (3.15), (3.10) and (3.9) we have

$$P_n \sim A\,\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\exp\left\{-\sum_{p\geq 2}(-1)^p\left(\frac{p-1}{2p(p+1)}-\frac{1}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k=n}^{\infty}\frac{1}{k^p}\right\}. \tag{3.16}$$

From (3.16) and Lemma 5 we have

$$\begin{aligned}P_n \sim{}& A\,\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\\ &\cdot\exp\left\{-\sum_{p\geq 2}(-1)^p\left(\frac{p-1}{2p(p+1)}-\frac{1}{12p}+\frac{B_{b+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!(-1)^k B_k}{(p-1)!k!}\frac{1}{n^{p-i+k}}\right\}.\end{aligned} \tag{3.17}$$

From (3.17) and Lemma 7 it follows that

$$P_n \sim A\,\frac{e^{n(n-1)/2}}{\sqrt{n!}}\left(\frac{e}{\sqrt{2\pi}}\right)^{n-1}\frac{1}{n^{\frac{1}{12}}}\exp\left\{-\sum_{p\geq 1}\left(\frac{B_{p+1}}{2p(p+1)}+\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}\right\}. \tag{3.18}$$

From (3.18), (3.14) and Lemma 1 it follows that

$$P_n \sim C^{-1} \frac{e^{n(n+2)/2}}{n^{(3n+2)/6}(2\pi)^{(2n+1)/4}} \exp\left\{-\sum_{p\geq 1} \frac{B_{p+1}+B_{p+2}}{p(p+1)} \frac{1}{n^p}\right\}$$

where

$$\begin{aligned} C &= \prod_{k=1}^{\infty} \left\{ k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \right\} \\ &= \lim_{n\to\infty} \frac{1}{(n+1)^{\frac{1}{12}}} \prod_{k=1}^{n} \left\{ k! \Big/ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \right\} \\ &= \lim_{n\to\infty} \frac{1}{n^{\frac{1}{12}}} \prod_{k=1}^{n} \left\{ k! \Big/ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \right\}. \end{aligned}$$

Thus we have proved our theorem.

4. Calculating the Constant C

We have

$$\begin{aligned} C &= \prod_{k=1}^{n} \left\{ k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \right\} \prod_{k=n+1}^{\infty} \left\{ k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \right\} \\ &= \frac{1}{(n+1)^{\frac{1}{12}}} \prod_{k=1}^{n} \left\{ k! \Big/ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \right\} \prod_{k=n+1}^{\infty} \left\{ k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \right\}, \end{aligned} \tag{4.1}$$

and we wish to estimate the second factor.
From Lemma 1,

$$k! \Big/ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \sim \exp\left\{ \sum_{p\geq 1} \frac{B_{p+1}}{p(p+1)} \frac{1}{k^p} \right\}$$

and

$$\left(1+\frac{1}{k}\right)^{\frac{1}{12}} = \exp\left\{ \sum_{p\geq 1} \frac{(-1)^{p+1}}{12p} \frac{1}{k^p} \right\}$$

it follows that

$$k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \sim \exp\left\{ \sum_{p\geq 2} \left(\frac{(-1)^p}{12p} + \frac{B_{p+1}}{p(p+1)} \right) \frac{1}{k^p} \right\} \tag{4.2}$$

and using Lemma 8,

$$\begin{aligned} &\prod_{k=n+1}^{\infty} \left\{ k! \Big/ \left\{ \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \left(1+\frac{1}{k}\right)^{\frac{1}{12}} \right\} \right\} \\ &\qquad \sim \exp\left\{ \sum_{p\geq 2} \left(\frac{(-1)^p}{12p} + \frac{B_{p+1}}{p(p+1)} \right) \sum_{k=n+1}^{\infty} \frac{1}{k^p} \right\} \end{aligned}$$

$$\sim \exp\left\{\sum_{p\geq 2}\left(\frac{(-1)^p}{12p}+\frac{B_{p+1}}{p(p+1)}\right)\sum_{k\geq 0}\frac{(p-2+k)!B_k}{(p-1)!k!}\cdot\frac{1}{n^{p-1+k}}\right\}$$

$$\sim \exp\left\{\sum_{p\geq 1}\left(\frac{(-1)^{p+1}}{12p}-\frac{B_{p+1}}{2p(p+1)}-\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}\right\}. \tag{4.3}$$

It follows that

$$C \sim \frac{1}{(n+1)^{\frac{1}{12}}}\prod_{k=1}^{n}\left\{k!\Big/\left\{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right\}\right\}\exp\left\{\sum_{p\geq 1}\left(\frac{(-1)^{p+1}}{12p}-\frac{B_{p+1}}{2p(p+1)}-\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}\right\}$$

$$= \frac{1}{n^{\frac{1}{12}}}\prod_{k=1}^{n}\left\{k!\Big/\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right\}\exp\left\{-\sum_{p\geq 1}\left(\frac{B_{p+1}}{2p(p+1)}+\frac{B_{p+2}}{p(p+1)}\right)\frac{1}{n^p}\right\}. \tag{4.4}$$

We can use this with a large value of n to compute an approximation to C. Thus, with 50 digits accuracy, $n = 5000$ and 20 terms of the series, I find

$$C \approx 1.0463350667705031809809506569777760\ldots\ .$$

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