ELEMENTARY ALGEBRA IN RAMANUJAN'S LOST NOTEBOOK

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Abstract. We prove a number of algebraic relations that appear in Ramanujan's Lost Notebook.

1. Introduction

In 1976, George E. Andrews discovered a manuscript which is essentially all we have of Ramanujan's work in the last year of his life, after he returned to India from England. Andrews dubbed this manuscript "Ramanujan's lost notebook". The story of this discovery, together with a few tantalizing details, is told in [1]. In 1988, to coincide with Ramanujan's birth centenary, a volume [3] was published in India, consisting of photocopies of the lost notebook, together with other unpublished work of Ramanujan.

Page 344 of [3] contains the following twelve statements.

If $g^4 = 5$ then

$$\frac{\sqrt[5]{3+2g} - \sqrt[5]{4-4g}}{\sqrt[5]{3+2g} + \sqrt[5]{4-4g}} = 2 + g + g^2 + g^3. \tag{1.1}$$

If $g^5 = 2$ then

$$\frac{\sqrt{g+3} + \sqrt{5g-5}}{\sqrt{g+3} - \sqrt{5g-5}} = g + g^2. \tag{1.2}$$

If $g^5 = 2$ then

$$\frac{\sqrt{g^2+1} + \sqrt{4g-3}}{\sqrt{g^2+1} - \sqrt{4g-3}} = \frac{1}{5}(1 + g^2 + g^3 + g^9)^2. \tag{1.3}$$

If $g^5 = 3$ then

$$\frac{\sqrt{g^2+1} + \sqrt{5g-5}}{\sqrt{g^2+1} - \sqrt{5g-5}} = \frac{1}{g} + g + g^2 + g^3. \tag{1.4}$$

If $g^5 = 2$ then

$$\sqrt{1+g^2} = \frac{g^4 + g^3 + g - 1}{\sqrt{5}}. \tag{1.5}$$

If $g^5 = 2$ then

$$\sqrt{4g-3} = \frac{g^9 + g^7 - g^6 - 1}{\sqrt{5}}. \tag{1.6}$$

If $g^5 = 3$ then

$$\sqrt[3]{2-g^3} = \frac{1+g-g^2}{\sqrt[3]{5}}. \tag{1.7}$$

If $g^5 = 2$ then

$$\sqrt[5]{1+g+g^3} = \frac{\sqrt{1+g^2}}{\sqrt[10]{5}}, \tag{1.8}$$

$$\sqrt[3]{\frac{1}{3}} + \sqrt[3]{\frac{5}{3}} = \sqrt{\frac{\sqrt[3]{5}-1}{2-\sqrt[3]{5}}\sqrt[3]{3}} = \sqrt[3]{\frac{3+\sqrt[3]{5}}{\sqrt[3]{5}-1}} = \sqrt[5]{\frac{3\sqrt[3]{3}+\sqrt[3]{15}}{2-\sqrt[3]{5}}}, \tag{1.9}$$

(here Ramanujan accidentally omitted the $\sqrt[3]{3}$ under the radical in the second expression),

$$\begin{aligned}\left\{\sqrt[3]{(a+b)(a^2+b^2)}-a\right\}&\left\{\sqrt[3]{(a+b)(a^2+b^2)}-b\right\}\\ &= \frac{\sqrt[3]{(a+b)^2}-\sqrt[3]{a^2+b^2}}{\sqrt[3]{(a+b)^2}+\sqrt[3]{a^2+b^2}}(a^2+ab+b^2)\end{aligned} \tag{1.10}$$

$$\frac{(\sqrt{a^2+ab+b^2}-a)(\sqrt{a^2+ab+b^2}-b)}{a+b-\sqrt{a^2+ab+b^2}} = a+b, \tag{1.11}$$

and

$$\begin{aligned}\left\{-c+\sqrt{(b+c)(c+a)}\right\}\left\{-a+\sqrt{(c+a)(a+b)}\right\}&\left\{-b+\sqrt{(a+b)(b+c)}\right\}\\ &= 2\left(\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{ab+bc+ca}\right)^{-2}.\end{aligned} \tag{1.12}$$

The object of this note is to provide as simple a proof as we can of each of these claims, and thereby shed a little light on how Ramanujan may have found them.

It should be remarked that B. C. Berndt, H. H. Chan and L–C. Zhang [2] give a detailed proof of (1.1) and indicate proofs of (1.2)–(1.9), in a much more far–reaching paper. But they do not mention (1.10)–(1.12).

2. The Proofs

We start with *componendo et dividendo*, which states that

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \text{ if and only if } \frac{a}{b} = \frac{c}{d}.$$

Proof of (1.1). We can write (1.1)

$$\frac{\sqrt[5]{2g+3}+\sqrt[5]{4g-4}}{\sqrt[5]{2g+3}-\sqrt[5]{4g-4}} = \frac{2+g+g^2+g^3}{1}. \tag{2.1}$$

If we apply *componendo et dividendo* to (2.1), we have

$$\sqrt[5]{\frac{2g+3}{4g-4}} = \frac{3+g+g^2+g^3}{1+g+g^2+g^3} = \frac{\dfrac{4}{g-1}+2}{\dfrac{4}{g-1}} = \frac{g+1}{2}. \tag{2.2}$$

We have

$$\begin{aligned}\left(\frac{g+1}{2}\right)^5 &= \frac{1}{32}(g^5+5g^4+10g^3+10g^2+5g+1)\\ &= \frac{1}{32}(5g+25+10g^3+10g^2+5g+1)\\ &= \frac{1}{32}(10g^3+10g^2+10g+26)\\ &= \frac{1}{32}(10(g^3+g^2+g+1)+16)\\ &= \frac{1}{32}\left(\frac{40}{g-1}+16\right) = \frac{1}{32}\frac{16g+24}{g-1} = \frac{2g+3}{4g-4},\end{aligned}$$

and (2.1) is established.

Proof of (1.2). Equation (1.2) can be written

$$\frac{\sqrt{g+3}+\sqrt{5g-5}}{\sqrt{g+3}-\sqrt{5g-5}} = \frac{g+g^2}{1}. \tag{2.3}$$

Thus, by *componendo et dividendo*, we see that we need to show that

$$\sqrt{\frac{g+3}{5g-5}} = \frac{1+g+g^2}{-1+g+g^2}. \tag{2.4}$$

Now,

$$\begin{aligned}\left(\frac{1+g+g^2}{-1+g+g^2}\right)^2 &= \frac{g^4+2g^3+3g^2+2g+1}{g^4+2g^3-g^2-2g+1}\\ &= \frac{(g^2+1)(g^4+2g^3+3g^2+2g+1)}{(g^2+1)(g^4+2g^3-g^2-2g+1)}\\ &= \frac{g^6+2g^5+4g^4+4g^3+4g^2+2g+1}{g^6+2g^5-2g+1}\\ &= \frac{2g+4+4g^4+4g^3+4g^2+2g+1}{2g+4-2g+1}\\ &= \frac{4g^4+4g^3+4g^2+4g+5}{5}\\ &= \frac{4(g^4+g^3+g^2+g+1)+1}{5}\\ &= \frac{1}{5}\left(\frac{4}{g-1}+1\right) = \frac{g+3}{5g-5},\end{aligned}$$

and (2.3) is proved.

Proof of (1.3). The right side of (1.3) is

$$\begin{aligned}\frac{1}{5}(1+g^2+g^3+g^9)^2 &= \frac{1}{5}(1+g^2+g^3+2g^4)^2\\ &= \frac{1}{5}(4g^8+4g^7+5g^6+2g^5+5g^4+2g^3+2g^2+1)\\ &= \frac{1}{5}(8g^3+8g^2+10g+4+5g^4+2g^3+2g^2+1)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{5}(5g^4 + 10g^3 + 10g^2 + 10g + 5) \\
&= g^4 + 2g^3 + 2g^2 + 2g + 1 \\
&= (g^4 + g^3 + g^2 + g + 1) + g^3 + g^2 + g \\
&= \frac{1}{g-1} + g^3 + g^2 + g = \frac{g^4 - g + 1}{g-1},
\end{aligned} \tag{2.5}
$$

so by *componendo et dividendo*, (1.3) is equivalent to

$$\sqrt{\frac{g^2+1}{4g-3}} = \frac{g^4}{g^4 - 2g + 2} = \frac{1}{1 - \frac{2}{g^3} + \frac{2}{g^4}} = \frac{1}{1 + g - g^2}, \tag{2.6}$$

or to

$$(g^2 - g - 1)^2 = \frac{4g-3}{g^2+1}. \tag{2.7}$$

Now,

$$
\begin{aligned}
(g^2+1)(g^2-g-1)^2 &= (g^2+1)(g^4 - 2g^3 - g^2 + 2g + 1) \\
&= g^6 - 2g^5 + 2g + 1 \\
&= 2g - 4 + 2g + 1 = 4g - 3,
\end{aligned}
$$

and (2.7) is proved.

Proof of (1.4). Equation (1.4) can be written

$$\frac{\sqrt{g^2+1} + \sqrt{5g-5}}{\sqrt{g^2+1} - \sqrt{5g-5}} = \frac{1 + g^2 + g^3 + g^4}{g}, \tag{2.8}$$

so by *componendo et dividendo*, (1.4) is equivalent to

$$\sqrt{\frac{g^2+1}{5g-5}} = \frac{1 + g + g^2 + g^3 + g^4}{1 - g + g^2 + g^3 + g^4} = \frac{\frac{2}{g-1}}{\frac{2}{g-1} - 2g} = \frac{1}{1 - g(g-1)} = \frac{1}{1 + g - g^2}, \tag{2.9}$$

or to

$$(g^2 - g - 1)^2 = \frac{5g-5}{g^2+1}. \tag{2.10}$$

Now,

$$
\begin{aligned}
(g^2+1)(g^2-g-1)^2 &= (g^2+1)(g^4 - 2g^3 - g^2 + 2g + 1) \\
&= g^6 - 2g^5 + 2g + 1 = 3g - 6 + 2g + 1 = 5g - 5,
\end{aligned}
$$

and (2.10) is proved.

Proof of (1.5). We have

$$
\begin{aligned}
(g^4 + g^3 + g - 1)^2 &= g^8 + 2g^7 + g^6 + 2g^5 - 2g^3 + g^2 - 2g + 1 \\
&= 2g^3 + 5g^2 + 2g + 4 - 2g^3 - 2g + 1 = 5(g^2+1),
\end{aligned}
$$

and (1.5) is proved.

Proof of (1.6). We have

$$(g^9 + g^7 - g^6 - 1)^2 = (2g^4 + 2g^2 - 2g - 1)^2$$

$$= 4g^8 + 8g^6 - 8g^5 - 8g^3 + 4g + 1 = 8g^3 + 16g - 16 - 8g^3 + 4g + 1 = 5(4g - 3),$$

and (1.6) is proved.

Proof of (1.7). We have

$$\begin{aligned}(1 + g - g^2)^3 &= (1 + g - g^2)(1 + 2g - g^2 - 2g^3 + g^4)\\ &= 1 + 3g - 5g^3 + 3g^5 - g^6\\ &= 1 + 3g - 5g^3 + 9 - 3g = 5(2 - g^3),\end{aligned}$$

and (1.7) is proved.

Proof of (1.8). We have

$$\begin{aligned}(1 + g + g^3)^2 &= g^6 + 2g^4 + 2g^3 + g^2 + 2g + 1\\ &= 2g + 2g^4 + 2g^3 + g^2 + 2g + 1\\ &= 2g^4 + 2g^3 + g^2 + 4g + 1\end{aligned}$$

while

$$\begin{aligned}(g^2 + 1)^5 &= g^{10} + 5g^8 + 10g^6 + 10g^4 + 5g^2 + 1\\ &= 4 + 10g^3 + 20g + 10g^4 + 5g^2 + 1\\ &= 10g^4 + 10g^3 + 5g^2 + 20g + 5,\end{aligned}$$

from which (1.8) follows.

Proof of (1.9). Ramanujan might have written (1.9) as follows.

If $g^3 = 5$ then

$$\frac{g+1}{\sqrt[3]{3}} = \sqrt{\frac{g-1}{2-g}\sqrt[3]{3}} = \sqrt[3]{\frac{3+g}{g-1}} = \sqrt[5]{\frac{3+g}{2-g}\sqrt[3]{3}}. \tag{2.11}$$

These three identities are equivalent to

$$\frac{(g+1)^2}{3} = \frac{g-1}{2-g}, \quad \frac{(g+1)^3}{3} = \frac{3+g}{g-1} \quad \text{and} \quad \frac{(g+1)^5}{9} = \frac{3+g}{2-g}, \tag{2.12}$$

and the third of these follows from the other two.

We have

$$(2 - g)(g + 1)^2 = (2 - g)(g^2 + 2g + 1) = -g^3 + 3g + 2 = -5 + 3g + 2 = 3g - 3$$

and

$$\begin{aligned}(g - 1)(g + 1)^3 &= (g - 1)(g^3 + 3g^2 + 3g + 1)\\ &= g^4 + 2g^3 - 2g - 1\\ &= 5g + 10 - 2g - 1 = 3g + 9\end{aligned}$$

and (1.9) is proved.

Proof of (1.10). Let $a + b = X^3$, $a^2 + b^2 = Y^6$. Then the left side of (1.10) is

$$\begin{aligned}(XY^2 - a)(XY^2 - b) &= X^2Y^4 - (a + b)XY^2 + ab\\ &= X^2Y^4 - X^4Y^2 + \frac{1}{2}(X^6 - Y^6)\\ &= \frac{1}{2}(X^6 - 2X^4Y^2 + 2X^2Y^4 - Y^6)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(X^2 - Y^2)(X^4 - X^2Y^2 + Y^4) \\
&= \frac{1}{2}(X^2 - Y^2)\frac{X^6 + Y^6}{X^2 + Y^2} \\
&= \frac{X^2 - Y^2}{X^2 + Y^2} \cdot \frac{1}{2}(X^6 + Y^6),
\end{aligned}$$

which is the right side of (1.10).

Proof of (1.11). Let $\sqrt{a^2 + ab + b^2} = X$. Then the left side of (1.11) is

$$\frac{(X-a)(X-b)}{a+b-X} = \frac{X^2 + ab - (a+b)X}{a+b-X} = \frac{(a+b)^2 - (a+b)X}{a+b-X},$$

which is the right side of (1.11).

Proof of (1.12). Let $\sqrt{(c+a)(a+b)} = X$, $\sqrt{(a+b)(b+c)} = Y$, $\sqrt{(b+c)(c+a)} = Z$. Then

$$X^2 - a^2 = Y^2 - b^2 = Z^2 - c^2 = ab + bc + ca$$

and

$$\begin{aligned}
&(X+a)(Y+b)(Z+c) \\
&= XYZ + cXY + aYZ + bZX + bcX + caY + abZ + abc \\
&= (a+b)(b+c)(c+a) + c(a+b)Z + a(b+c)X + b(c+a)Y + bcX + caY + abZ + abc \\
&= (ab+bc+ca)(X+Y+Z) + (ab+bc+ca)(a+b+c) \\
&= (ab+bc+ca)(X+Y+Z+a+b+c).
\end{aligned}$$

It follows that

$$\begin{aligned}
(X-a)(Y-b)(Z-c) &= \frac{ab+bc+ca}{X+a} \cdot \frac{ab+bc+ca}{Y+b} \cdot \frac{ab+bc+ca}{Z-c} \\
&= \frac{(ab+bc+ca)^3}{(X+a)(Y+b)(Z+c)} \\
&= \frac{(ab+bc+ca)^2}{X+Y+Z+a+b+c} \\
&= \frac{2(ab+bc+ca)^2}{2(X+Y+Z+a+b+c)} \\
&= \frac{2(ab+bc+ca)^2}{(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a})^2},
\end{aligned}$$

and (1.12) is proved!

References

[1] G. E. Andrews, *An introduction to Ramanujan's "lost" notebook*, Amer. Math. Monthly, **86.2** (1979), 89–108.

[2] B. C. Berndt, H. H. Can, and L–C. Zhang, *Radicals and units in Ramanujan's work*, Acta Arith., LXXXVII (1998–1999), 145–158.

[3] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

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