VARIANTS OF THE FILBERT MATRIX

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Abstract. A variation of the Filbert matrix from [1] is introduced, which has one additional Fibonacci factor in the numerator. We also introduce its Lucas counterpart by taking Lucas numbers instead of Fibonacci numbers in a similar manner. Explicit formulas are derived for the LU-decompositions, their inverses, the inverse matrix, as well as the Cholesky decompositions. The approach is to use q-analysis and to guess the relevant quantities, and proving them later by induction.

1. Introduction

The Filbert matrix $H_n = \left(\check{h}_{ij}\right)_{i,j=1}^n$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [4].

After the Filbert matrix, several generalizations and analogues of it have been investigated and studied by several authors. For the readers convenience, we briefly summarize these:

- In [1], Kılıç and Prodinger studied the generalized Filbert Matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter.
- After this generalization, Prodinger [3] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$.
- Recently, in [2], Kılıç and Prodinger gave a further generalization of the generalized Filbert Matrix $\mathcal{F}$ by defining the matrix $\mathcal{Q}$ with entries h_{ij} as follows
$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$
where $r \geq -1$ and $k \geq 1$ are integer parameters.

In the works summarized above, the authors derived explicit formulas for the LU-decomposition, their inverses, and the Cholesky factorization.

In this paper, we introduce two new variations of the Filbert matrix H_n, and define the matrices $\mathcal{G}$ and $\mathcal{L}$ with entries g_{ij} and t_{ij} by

$$g_{ij} = \frac{F_{\lambda(i+j)+r}}{F_{\lambda(i+j)+s}} \quad \text{and} \quad t_{ij} = \frac{L_{\lambda(i+j)+r}}{L_{\lambda(i+j)+s}}$$

where s, r and λ are integer parameters such that $s \neq r$, and $s \geq -1$ and $\lambda \geq 1$. This is the first nontrivial instance where the numerator of the entries is *not* equal to zero.

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Our approach will be as follows. We will use the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad L_n = \alpha^n + \beta^n = \alpha^n\left(1+q^n\right)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

Throughout this paper we will use the notation of the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$.

We rewrite the entries of the matrices $\mathcal{G}$ and $\mathcal{L}$ in terms of the q-Pochhammer symbol:

$$g_{ij} = \mathbf{i}^{r-s}q^{-\frac{1}{2}(r-s)}\frac{1-q^{\lambda(i+j)+r}}{1-q^{\lambda(i+j)+s}} \quad \text{and} \quad t_{ij} = \mathbf{i}^{r-s}q^{-\frac{1}{2}(r-s)}\frac{1+q^{\lambda(i+j)+r}}{1+q^{\lambda(i+j)+s}}.$$

We will derive explicit formulas for the LU-decompositions of matrices $\mathcal{G}_N$ and $\mathcal{L}_N$, and their inverses. Similarly to the results of [1, 2], the size of the matrices does not really matter, and they can be thought as infinite matrices $\mathcal{G}$, $\mathcal{L}$ and we may restrict it whenever necessary to the first N rows, resp. columns and write $\mathcal{G}_N$ and $\mathcal{L}_N$. We also provide the Cholesky decompositions. All the identities we will obtain hold for general q, and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

First, we will present all the results related to the matrix $\mathcal{G}$. Second, we will give all the results related to the matrix $\mathcal{L}$. Finally we will indicate some proofs related to the matrix $\mathcal{G}$.

As an illustration, we always write out the Fibonacci/Lucas case explicitly for $\lambda = 2$.

2. Results for $\mathcal{G}$

We obtain the LU-decomposition $\mathcal{G} = L \cdot U$.

Theorem 2.1. *For $1 \le d \le n$ we have*

$$L_{n,d} = \frac{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda(d+1)+s};q^\lambda)_d}{(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}(q^{\lambda(n+1)+s};q^\lambda)_d}\frac{1-q^{\lambda(d^2+n)+sd-s+r}}{1-q^{\lambda(d^2+d)+sd-s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.1. *For $1 \le d \le n$,*

$$L_{n,d} = \left(\prod_{t=d}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d} F_{2(t+d)+s}\right)\left(\prod_{t=1}^{n-d} F_{2t}\right)^{-1}\left(\prod_{t=1}^{d} F_{2(t+n)+s}\right)^{-1}\frac{F_{2(d^2+n)+sd-s+r}}{F_{2(d^2+d)+sd-s+r}}.$$

Theorem 2.2. *For $1 \le d \le n$ we have*

$$U_{d,n} = \frac{\mathbf{i}^{r-s+2}q^{-\frac{3s}{2}+\frac{r}{2}+sd+\lambda(d^2-d)}(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-1}}{(q^\lambda;q^\lambda)_{n-d}(q^{\lambda(n+1)+s};q^\lambda)_d(q^{\lambda(d+1)+s};q^\lambda)_{d-1}}\frac{1-q^{\lambda(d^2+n)+sd-s+r}}{1-q^{\lambda(d^2-d)+sd-2s+r}}(1-q^{s-r}).$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.2. *For $1 \le d \le n$*

$$U_{d,n} = \frac{(-1)^{r+ds+1}\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d-1} F_{2t}\right)F_{2(d^2+n)+sd-s+r}F_{s-r}}{\left(\prod_{t=1}^{n-d} F_{2t}\right)\left(\prod_{t=1}^{d} F_{2(n+t)+s}\right)\left(\prod_{t=1}^{d-1} F_{2(d+t)+s}\right)F_{2(d^2-d)+sd-2s+r}}.$$

We could also determine the inverses of the matrices L and U.

Theorem 2.3. *For* $1 \le d \le n$ *we have*

$$L_{n,d}^{-1} = \frac{q^{\lambda\frac{n(n-1)}{2}-\lambda dn+\lambda\frac{d(d+1)}{2}}(q^\lambda;q^\lambda)_{n-1}(q^{\lambda(d+1)+s};q^\lambda)_{n-1}(-1)^{n-d}}{(q^\lambda;q^\lambda)_{d-1}(q^{\lambda(n+1)+s};q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{n-d}}\frac{1-q^{\lambda(n^2-d)+sn-2s+r}}{1-q^{\lambda(n^2-n)+sn-2s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.3. *For* $1 \le d \le n$

$$L_{n,d}^{-1} = \frac{(-1)^{n-d}\Big(\prod_{t=1}^{n-1}F_{2t}\Big)\Big(\prod_{t=1}^{n-1}F_{2(d+t)+s}\Big)F_{2(n^2-d)+sn-2s+r}}{\Big(\prod_{t=1}^{d-1}F_{2t}\Big)\Big(\prod_{t=1}^{n-1}F_{2(n+t)+s}\Big)\Big(\prod_{t=1}^{n-d}F_{2t}\Big)F_{2(n^2-n)+sn-2s+r}}.$$

Theorem 2.4. *For* $1 \le d \le n$ *we have*

$$U_{d,n}^{-1} = \frac{q^{-\lambda\frac{n(n-1)}{2}-\lambda dn+\lambda\frac{d(d+1)}{2}+\frac{3s-r}{2}-sn}(q^{\lambda(n+1)+s};q^\lambda)_{d-1}(q^{\lambda(d+1)+s};q^\lambda)_{2n-d}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}}$$
$$\times\frac{1-q^{\lambda(n^2-d)+sn-2s+r}}{1-q^{\lambda(n^2+n)+sn-s+r}}\frac{1}{1-q^{s-r}}\mathbf{i}^{s-r}(-1)^{n-d-1},$$

and its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.4. *For* $1 \le d \le n$

$$U_{d,n}^{-1} = \frac{(-1)^{n-d-r-ns+1}\Big(\prod_{t=1}^{d-1}F_{2(n+t)+s}\Big)\Big(\prod_{t=1}^{2n-d}F_{2(d+t)+s}\Big)F_{2(n^2-d)+sn-2s+r}}{\Big(\prod_{t=1}^{n-1}F_{2t}\Big)\Big(\prod_{t=1}^{n-d}F_{2t}\Big)\Big(\prod_{t=1}^{d-1}F_{2t}\Big)F_{2(n^2+n)+sn-s+r}F_{s-r}}.$$

As a consequence, we can compute the determinant of $\mathcal{G}_n$, since it is simply evaluated as $U_{1,1}\cdots U_{n,n}$.

Theorem 2.5.

$$\det\mathcal{G}_n = (-1)^n\mathbf{i}^{n(r-s)}q^{\frac{1}{6}n(n+1)(3s+2\lambda(n-1))+\frac{-3sn+rn}{2}}(1-q^{s-r})^n$$
$$\times\prod_{d=1}^{n}\frac{(q^\lambda;q^\lambda)_{d-1}^2}{(q^{\lambda(d+1)+s};q^\lambda)_d(q^{\lambda(d+1)+s};q^\lambda)_{d-1}}\frac{1-q^{\lambda(d^2+d)+sd-s+r}}{1-q^{\lambda(d^2-d)+sd-2s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.5.

$$\det\mathcal{G}_n = (-1)^{n(r+1)+\binom{n+1}{2}s}F_{s-r}^n\prod_{d=1}^{n}\frac{F_{2(d^2+d)+sd-s+r}}{F_{4d+s}F_{2(d^2-d)+sd-2s+r}}\prod_{t=1}^{d-1}\frac{F_{2t}^2}{F_{2(d+t)+s}^2}.$$

Now we compute the inverse of the matrix $\mathcal{G}$. This time it depends on the dimension, so we compute $(\mathcal{G}_N)^{-1}$.

Theorem 2.6. *For* $1 \le n, d \le N$:

$$(\mathcal{G}_N)_{n,d}^{-1} = \frac{(q^{\lambda(d+1)+s};q^\lambda)_N(q^{\lambda(n+1)+s};q^\lambda)_N}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{N-d}}$$
$$\times\frac{\mathbf{i}^{s-r}(-1)^{n-d-1}q^{\lambda\frac{d(d+1)}{2}+\lambda\frac{n(n+1)}{2}-\lambda Nd-\lambda Nn+\frac{3s-r}{2}-sN}}{(1-q^{\lambda(n+d)+s})(1-q^{s-r})}\frac{1-q^{\lambda(N^2+N-n-d)+sN-2s+r}}{1-q^{\lambda(N^2+N)+sN-s+r}}.$$

Remark. The inverse matrix was *not* computed using the inverses of L and U, but rather obtained directly by our usual guessing strategy. While the first alternative would mean that we would have to simplify a sum, the second approach stays within our chosen method. This remark applies as well to the Lucas case that is discussed in the next section.

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.6. *For $1 \le n, d \le N$:*

$$(\mathcal{G}_N)^{-1}_{n,d} = \frac{\left(\prod_{t=1}^{N} F_{2(d+t)+s}\right)\left(\prod_{t=1}^{N} F_{2(n+t)+s}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{N-n} F_{2t}\right)\left(\prod_{t=1}^{N-d} F_{2t}\right)} \frac{(-1)^{n-d-r+1} F_{2(N^2+N-n-d)+sN-2s+r}}{F_{2(n+d)+s} F_{s-r} F_{2(N^2+N)+sN-s+r}}.$$

Finally, we provide the Cholesky decomposition.

Theorem 2.7. *For $i, j \ge 1$:*

$$\mathcal{C}_{n,d} = \frac{\mathbf{i}^{\frac{r-s}{2}+1} q^{\lambda\frac{d(d-1)}{2} + \frac{-3s+r}{4} + \frac{sd}{2}} (q^\lambda; q^\lambda)_{n-1}}{(q^{\lambda(n+1)+s}; q^\lambda)_d (q^\lambda; q^\lambda)_{n-d}} (1 - q^{\lambda(d^2+n)+sd-s+r}) \times \sqrt{\frac{(1-q^{2\lambda d+s})(1-q^{s-r})}{(1-q^{\lambda(d^2+d)+sd-s+r})(1-q^{\lambda(d^2-d)+sd-2s+r})}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 2.7. *For $i, j \ge 1$:*

$$\mathcal{C}_{n,d} = \mathbf{i}^{r+s(d-2)+1} \left(\prod_{t=n-d+1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d} F_{2(n+t)+s}\right)^{-1} \times F_{2(d^2+n)+sd-s+r} \sqrt{\frac{F_{4d+s} F_{s-r}}{F_{2(d^2+d)+sd-s+r} F_{2(d^2-d)+sd-2s+r}}}.$$

3. Results for $\mathcal{L}$

Now we collect our results related to the matrix $\mathcal{L}$.

For convenience, we use the same letters L, U, $\mathcal{C}$, but with the different meaning.

We obtain the LU-decomposition $\mathcal{L} = L \cdot U$.

Theorem 3.1. *For $1 \le d \le n$ we have*

$$L_{n,d} = \frac{(q^\lambda; q^\lambda)_{n-1} (-q^{\lambda(d+1)+s}; q^\lambda)_d}{(q^\lambda; q^\lambda)_{n-d} (q^\lambda; q^\lambda)_{d-1} (-q^{\lambda(n+1)+s}; q^\lambda)_d} \frac{1-(-1)^d q^{\lambda(d^2+n)+sd-s+r}}{1-(-1)^d q^{\lambda(d^2+d)+sd-s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.1. *For $1 \le d \le n$,*

$$L_{n,d} = \frac{\left(\prod_{t=d}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d} L_{2(t+d)+s}\right)}{\left(\prod_{t=1}^{n-d} F_{2t}\right)\left(\prod_{t=1}^{d} L_{2(t+n)+s}\right)} \times \begin{cases} \frac{F_{2(d^2+n)+sd-s+r}}{F_{2(d^2+d)+sd-s+r}} & \text{if } d \text{ is even,} \\ \frac{L_{2(d^2+n)+sd-s+r}}{L_{2(d^2+d)+sd-s+r}} & \text{if } d \text{ is odd.} \end{cases}$$

Theorem 3.2. *For $1 \le d \le n$ we have*

$$U_{d,n} = \frac{q^{\frac{r-3s}{2}+\lambda d(d-1)+ds}\mathbf{i}^{r-s}(-1)^d(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-1}}{(q^\lambda;q^\lambda)_{n-d}(-q^{\lambda(d+1)+s};q^\lambda)_{d-1}(-q^{\lambda(n+1)+s};q^\lambda)_d} \times \frac{1-(-1)^d q^{\lambda(d^2+n)+sd-s+r}}{1+(-1)^d q^{\lambda(d^2-d)+sd-2s+r}}(1-q^{s-r}).$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.2. *For $1 \le d \le n$*

$$U_{d,n} = \frac{(-1)^{r+d(s+1)}\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{d-1} F_{2t}\Big)F_{s-r}}{\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{d} L_{2(n+t)+s}\Big)\Big(\prod_{t=1}^{d-1} L_{2(d+t)+s}\Big)} \times \begin{cases} \dfrac{5^d F_{2(d^2+n)+sd-s+r}}{L_{2(d^2-d)+sd-2s+r}} & \text{if } d \text{ is even,} \\ \dfrac{5^{d-1} L_{2(d^2+n)+sd-s+r}}{F_{2(d^2-d)+sd-2s+r}} & \text{if } d \text{ is odd.} \end{cases}$$

We could also determine the inverses of the matrices L and U.

Theorem 3.3. *For $1 \le d \le n$ we have*

$$L^{-1}_{n,d} = \frac{q^{\lambda\frac{n(n-1)}{2}-\lambda dn+\lambda\frac{d(d+1)}{2}}(q^\lambda;q^\lambda)_{n-1}(-q^{\lambda(d+1)+s};q^\lambda)_{n-1}(-1)^{n-d}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(-q^{\lambda(n+1)+s};q^\lambda)_{n-1}} \times \frac{1+(-1)^n q^{\lambda(n^2-d)+sn-2s+r}}{1+(-1)^n q^{\lambda(n^2-n)+sn-2s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.3. *For $1 \le d \le n$*

$$L^{-1}_{n,d} = \frac{(-1)^{d+n}\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} L_{2(d+t)+s}\Big)}{\Big(\prod_{t=1}^{d-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{n-1} L_{2(n+t)+s}\Big)} \times \begin{cases} \dfrac{F_{2(n^2-d)+sn-2s+r}}{F_{2(n^2-n)+sn-2s+r}} & \text{if } n \text{ is odd,} \\ \dfrac{L_{2(n^2-d)+sn-2s+r}}{L_{2(n^2-n)+sn-2s+r}} & \text{if } n \text{ is even.} \end{cases}$$

Theorem 3.4. *For $1 \le d \le n$ we have*

$$U^{-1}_{d,n} = \frac{\mathbf{i}^{s-r}(-1)^d(-q^{\lambda(d+1)+s};q^\lambda)_{n-1}(-q^{\lambda(n+1)+s};q^\lambda)_n}{(1-q^{s-r})(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{n-1}} \times \frac{1+(-1)^n q^{\lambda(n^2-d)+sn-2s+r}}{1-(-1)^n q^{\lambda n(n+1)+sn-s+r}} q^{\frac{3s-r}{2}-\lambda\frac{n(n-1)}{2}+\lambda\frac{d(d+1)}{2}-ns-\lambda dn},$$

and its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.4. *For $1 \le d \le n$*

$$U^{-1}_{d,n} = \frac{(-1)^{r+d-ns}\Big(\prod_{t=1}^{n-1} L_{2(t+d)+s}\Big)\Big(\prod_{t=1}^{n} L_{2(t+n)+s}\Big)}{\Big(\prod_{t=1}^{n-1} F_{2t}\Big)\Big(\prod_{t=1}^{n-d} F_{2t}\Big)\Big(\prod_{t=1}^{d-1} F_{2t}\Big)F_{s-r}} \times \begin{cases} \dfrac{F_{2(n^2-d)+sn-2s+r}}{5^{n-1}L_{2(n^2+n)+sn-s+r}} & \text{if } n \text{ is odd,} \\ \dfrac{L_{2(n^2-d)+sn-2s+r}}{5^n F_{2(n^2+n)+sn-s+r}} & \text{if } n \text{ is even.} \end{cases}$$

As a consequence, we can compute the determinant of $\mathcal{L}_n$, since it is simply evaluated as $U_{1,1}\cdots U_{n,n}$ (we only state the Fibonacci version for $\lambda = 2$).

Theorem 3.5.

$$\det \mathcal{L}_n = 5^{\frac{1}{2}n(n+(-1)^d)}(-1)^{nr}\mathbf{i}^{n(s+1)(n+1)}F_{s-r}^n \prod_{d=1}^{n} \frac{1}{L_{4d+s}} \prod_{t=1}^{d-1} \frac{F_{2t}^2}{L_{2(d+t)+s}^2}$$
$$\times \begin{cases} \dfrac{F_{2(d^2+d)+sd-s+r}}{L_{2(d^2-d)+sd-2s+r}} & \text{if } d \text{ is even,} \\ \dfrac{L_{2(d^2+d)+sd-s+r}}{F_{2(d^2-d)+sd-2s+r}} & \text{if } d \text{ is odd.} \end{cases}$$

Now we compute the inverse of the matrix $\mathcal{L}$. This time it depends on the dimension, so we compute $(\mathcal{L}_N)^{-1}$.

Theorem 3.6. *For $1 \le d \le n \le N$:*

$$(\mathcal{L}_N)^{-1}_{n,d} = \frac{(-q^{\lambda(n+1)+s};q^\lambda)_N(-q^{\lambda(d+1)+s};q^\lambda)_N}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{N-d}}$$
$$\times \frac{\mathbf{i}^{s-r}(-1)^{n-d+N}q^{\lambda\frac{d(d+1)}{2}+\lambda\frac{n(n+1)}{2}-\lambda Nd-\lambda Nn+\frac{3s-r}{2}-sN}}{(1-q^{s-r})(1+q^{\lambda(n+d)+s})} \frac{1+(-1)^N q^{\lambda(N^2+N-n-d)+sN-2s+r}}{1-(-1)^N q^{\lambda(N^2+N)+sN-s+r}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.5. *For $1 \le d \le n \le N$:*

$$(\mathcal{L}_N)^{-1}_{n,d} = \frac{\left(\prod_{t=1}^{N} L_{2(d+t)+s}\right)\left(\prod_{t=1}^{N} L_{2(n+t)+s}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{N-n} F_{2t}\right)\left(\prod_{t=1}^{N-d} F_{2t}\right)} \frac{1}{F_{s-r}L_{2(n+d)+s}}$$
$$\times \begin{cases} \dfrac{L_{2(n^2-d)+sn-2s+r}}{5^N F_{2(n^2+n)+sn-s+r}} & \text{if } N \text{ is even,} \\ \dfrac{F_{2(N^2+N-n-d)+sN-2s+r}}{5^{N-1}L_{2(N^2+N)+sN-s+r}} & \text{if } N \text{ is odd.} \end{cases}$$

Finally, we provide the Cholesky decomposition.

Theorem 3.7. *For $n, d \ge 1$:*

$$\mathcal{C}_{n,d} = \frac{(1-(-1)^d q^{\lambda(d^2+n)+sd-s+r})\left(q^\lambda;q^\lambda\right)_{n-1}}{(q^\lambda;q^\lambda)_{n-d}\left(-q^{\lambda(n+1)+s};q^\lambda\right)_d}$$
$$\times \mathbf{i}^{-\lambda d(d-1)+2+\frac{r-s}{2}-d} q^{\lambda\frac{d(d-1)}{2}+\frac{-3s+r}{4}+\frac{sd}{2}}$$
$$\times \sqrt{\frac{(1+q^{2\lambda d+s})(1-q^{s-r})}{(1-(-1)^d q^{\lambda(d^2+d)+sd-s+r})(1+(-1)^d q^{\lambda(d^2-d)+sd-2s+r})}}.$$

Its Fibonacci Corollary for $\lambda = 2$ follows.

Corollary 3.6. *For $n, d \geq 1$:*

$$\mathcal{C}_{n,d} = \mathbf{i}^{r+ds}(-1)^{s+1}5^{d/2}\left(\prod_{t=n-d+1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d} L_{2(n+t)+s}\right)^{-1}\sqrt{L_{4d+s}F_{s-r}}$$
$$\times\begin{cases} F_{2(d^2+n)+sd-s+r}\dfrac{1}{\sqrt{F_{2(d^2+d)+sd-s+r}L_{2(d^2-d)+sd-2s+r}}} & \text{if } d \text{ is even,} \\[2ex] L_{2(d^2+n)+sd-s+r}\dfrac{1}{\sqrt{L_{2(d^2+d)+sd-s+r}F_{2(d^2-d)+sd-2s+r}}} & \text{if } d \text{ is odd.} \end{cases}$$

4. Proofs

We start with an introductory remark. For all the identities that we need to prove, experiments indicate that they are Gosper-summable. However, the entries that we encounter in our instances do not qualify for the q-Zeilberger algorithm that we used in our earlier papers. Therefore, it was necessary to *guess* the relevant quantities; the justification is then complete routine. However, this guessing procedure is (with all the parameters involved) extremely time consuming, and so we confined ourselves to the demonstration of two such proofs. We hope that extensions of the q-Zeilberger algorithm will be developed that fit our needs.

First, we show that $\sum_j L_{m,j}U_{j,n}$ is indeed the matrix $\mathcal{G}$, that is,

$$\sum_{1\leq d\leq \min\{m,n\}} L_{m,d}U_{d,n} = \mathbf{i}^{r-s}q^{-\frac{1}{2}(r-s)}\frac{1-q^{\lambda(m+n)+r}}{1-q^{\lambda(m+n)+s}}.$$

Since the formula is symmetric in m and n, we can assume without loss of generality that $m \geq n$.

However we have in fact a more general formula:

$$\sum_{K\leq d\leq n} L_{m,d}U_{d,n} = \mathbf{i}^{r-s}q^{-\frac{1}{2}(r-s)}$$
$$\times q^{\lambda K^2-\lambda K+sK-s}\frac{1-q^{\lambda K^2-\lambda K+\lambda m+\lambda n+r+sK-s}}{1-q^{\lambda K^2-\lambda K+r+sK-2s}}\frac{1-q^{r-s}}{1-q^{\lambda m+\lambda n+s}}$$
$$\times\frac{(q^\lambda;q^\lambda)_{m-1}(q^\lambda;q^\lambda)_{n-1}(q^s;q^\lambda)_{m+1}(q^s;q^\lambda)_{n+1}}{(q^\lambda;q^\lambda)_{m-K}(q^\lambda;q^\lambda)_{n-K}(q^s;q^\lambda)_{m+K}(q^s;q^\lambda)_{n+K}}.$$

The formula we need follows from setting $K := 1$.

We use (backward) induction to prove the more general formula. Clearly it is true for $K = n$, and the induction step amounts to show that

$$L_{m,K}U_{K,n} + \mathbf{i}^{r-s}q^{-\frac{1}{2}(r-s)}q^{\lambda(K+1)^2-\lambda(K+1)+s(K+1)-s}$$
$$\times\frac{1-q^{\lambda(K+1)^2-\lambda(K+1)+\lambda m+\lambda n+r+s(K+1)-s}}{1-q^{\lambda(K+1)^2-\lambda(K+1)+r+s(K+1)-2s}}\frac{1-q^{r-s}}{1-q^{\lambda m+\lambda n+s}}$$
$$\times\frac{(q^\lambda;q^\lambda)_{m-1}(q^\lambda;q^\lambda)_{n-1}(q^s;q^\lambda)_{m+1}(q^s;q^\lambda)_{n+1}}{(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_{n-K-1}(q^s;q^\lambda)_{m+(K+1)}(q^s;q^\lambda)_{n+(K+1)}}$$

$$\begin{aligned}
&= \mathbf{i}^{r-s} q^{-\frac{1}{2}(r-s)} q^{\lambda K^2-\lambda K+sK-s} \\
&\quad \times \frac{1-q^{\lambda K^2-\lambda K+\lambda m+\lambda n+r+sK-s}}{1-q^{\lambda K^2-\lambda K+r+sK-2s}} \frac{1-q^{r-s}}{1-q^{\lambda m+\lambda n+s}} \\
&\quad \times \frac{(q^\lambda;q^\lambda)_{m-1}(q^\lambda;q^\lambda)_{n-1}(q^s;q^\lambda)_{m+1}(q^s;q^\lambda)_{n+1}}{(q^\lambda;q^\lambda)_{m-K}(q^\lambda;q^\lambda)_{n-K}(q^s;q^\lambda)_{m+K}(q^s;q^\lambda)_{n+K}}.
\end{aligned}$$

By the definition of the matrices $L_{m,n}$ and $U_{m,n}$, the above equation takes the following form:

$$\begin{aligned}
&\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(K+1)+s};q^\lambda)_K}{(q^\lambda;q^\lambda)_{m-K}(q^\lambda;q^\lambda)_{K-1}(q^{\lambda(m+1)+s};q^\lambda)_K} \frac{1-q^{\lambda(K^2+m)+sK-s+r}}{1-q^{\lambda(K^2+K)+sK-s+r}} \\
&\quad \times \frac{q^{-\frac{3s}{2}+\frac{r}{2}+sK+\lambda(K^2-K)}(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{K-1}(1-q^{s-r})}{\mathbf{i}^{s-r-2}(q^\lambda;q^\lambda)_{n-K}(q^{\lambda(n+1)+s};q^\lambda)_K(q^{\lambda(K+1)+s};q^\lambda)_{K-1}} \\
&\quad \times \frac{1-q^{\lambda(K^2+n)+sK-s+r}}{1-q^{\lambda(K^2-K)+sK-2s+r}} \\
&\quad + \mathbf{i}^{r-s} q^{-\frac{1}{2}(r-s)} q^{\lambda(K+1)^2-\lambda(K+1)+s(K+1)-s} \\
&\quad \times \frac{1-q^{\lambda(K+1)^2-\lambda(K+1)+\lambda m+\lambda n+r+s(K+1)-s}}{1-q^{\lambda(K+1)^2-\lambda(K+1)+r+s(K+1)-2s}} \frac{(1-q^{r-s})}{1-q^{\lambda m+\lambda n+s}} \\
&\quad \times \frac{(q^\lambda;q^\lambda)_{m-1}(q^\lambda;q^\lambda)_{n-1}(q^s;q^\lambda)_{m+1}(q^s;q^\lambda)_{n+1}}{(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_{n-K-1}(q^s;q^\lambda)_{m+(K+1)}(q^s;q^\lambda)_{n+(K+1)}} \\
&= \mathbf{i}^{r-s} q^{-\frac{1}{2}(r-s)} q^{\lambda K^2-\lambda K+sK-s} \\
&\quad \times \frac{1-q^{\lambda K^2-\lambda K+\lambda m+\lambda n+r+sK-s}}{1-q^{\lambda K^2-\lambda K+r+sK-2s}} \frac{1-q^{r-s}}{1-q^{\lambda m+\lambda n+s}} \\
&\quad \times \frac{(q^\lambda;q^\lambda)_{m-1}(q^\lambda;q^\lambda)_{n-1}(q^s;q^\lambda)_{m+1}(q^s;q^\lambda)_{n+1}}{(q^\lambda;q^\lambda)_{m-K}(q^\lambda;q^\lambda)_{n-K}(q^s;q^\lambda)_{m+K}(q^s;q^\lambda)_{n+K}},
\end{aligned}$$

which, after some simplifications using the definition of the q-Pochhammer symbol, is equivalent to

$$\begin{aligned}
&(1-q^{s+2K\lambda})(1-q^{\lambda(K^2+m)+sK-s+r})(1-q^{\lambda m+\lambda n+s})(1-q^{\lambda(K^2+n)+sK-s+r}) \\
&\quad + q^{s+2K\lambda}(1-q^{r+K\lambda+m\lambda+n\lambda+K^2\lambda+Ks})(1-q^{\lambda(m-K)}) \\
&\quad \times (1-q^{\lambda(n-K)})(1-q^{\lambda K^2-\lambda K+r+sK-2s}) \\
&= (1-q^{\lambda K^2-\lambda K+\lambda m+\lambda n+r+sK-s})(1-q^{\lambda(m+K)+s}) \\
&\quad \times (1-q^{\lambda(n+K)+s})(1-q^{r-s+K\lambda+K^2\lambda+Ks})
\end{aligned}$$

and further to

$$\begin{aligned}
&-\frac{1}{q^{2s+K\lambda}}(q^{s+K\lambda}-q^{\lambda K^2+sK+r+m\lambda+n\lambda})(q^{r+K\lambda+K^2\lambda+Ks}-q^s) \\
&\qquad\qquad = (1-q^{\lambda K^2-\lambda K+\lambda m+\lambda n+r+sK-s})(1-q^{r-s+K\lambda+K^2\lambda+Ks}),
\end{aligned}$$

which is true by direct expansion. Thus the proof is completed.

Second, now we deal with

$$\sum_{n\le d\le m} L_{m,d} L^{-1}_{d,n}$$

and prove that it is 1 for $n = m$ (there is only one term in the sum) and 0 for $n > m$ since we have lower triangular matrices. So let us assume $m > n$. We will prove a general formula depending on an extra variable K:

$$\sum_{n\le d\le K} L_{m,d}L^{-1}_{d,n} = q^{\lambda\frac{K(K+1)}{2}+\lambda\frac{n(n-1)}{2}-\lambda nK}(-1)^{K-n}\frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_{K-n}}$$
$$\times\frac{(q^s;q^\lambda)_{n+K+1}(q^s;q^\lambda)_{m+1}}{(q^s;q^\lambda)_{n+1}(q^s;q^\lambda)_{m+K+1}}\frac{(1-q^{\lambda m+\lambda K^2+sK+\lambda K-s+r-\lambda n})}{(1-q^{\lambda m-\lambda n})(1-q^{\lambda K^2+sK+\lambda K-s+r})}.$$

The formula we need follows from setting $K := m$. Note that the RHS of the formula equals 0 when $K = m > n$ because of the term $(q^\lambda;q^\lambda)_{m-K-1}$ that appears in the denominator. The proof of the formula is by induction. Clearly it is true for $K = n$, and the induction step amounts to show that

$$\sum_{n\le d\le K} L_{m,d}L^{-1}_{d,n} + L_{m,K+1}L^{-1}_{K+1,n} = \sum_{n\le d\le K+1} L_{m,d}L^{-1}_{d,n},$$

which equals

$$\frac{(q^\lambda;q^\lambda)_{m-1}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_{K-n}}\frac{(q^s;q^\lambda)_{n+K+1}(q^s;q^\lambda)_{m+1}}{(q^s;q^\lambda)_{n+1}(q^s;q^\lambda)_{m+K+1}}$$
$$\times\frac{(1-q^{\lambda m+\lambda K^2+sK+\lambda K-s+r-\lambda n})}{(1-q^{\lambda m-\lambda n})(1-q^{\lambda K^2+sK+\lambda K-s+r})}q^{\lambda\frac{K(K+1)}{2}+\lambda\frac{n(n-1)}{2}-\lambda nK}(-1)^{K-n}$$
$$+\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(K+2+s};q^\lambda)_{K+1}}{(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_K(q^{\lambda(m+1)+s};q^\lambda)_{K+1}}$$
$$\times\frac{(q^\lambda;q^\lambda)_K(q^{\lambda(n+1)+s};q^\lambda)_K}{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda(K+2+s};q^\lambda)_K(q^\lambda;q^\lambda)_{K+1-n}}$$
$$\times\frac{1-q^{\lambda((K+1)^2-n)+s(K+1)-2s+r}}{1-q^{\lambda((K+1)^2-(K+1))+s(K+1)-2s+r}}\frac{1-q^{\lambda((K+1)^2+m)+s(K+1)-s+r}}{1-q^{\lambda((K+1)^2+(K+1))+s(K+1)-s+r}}$$
$$\times q^{\lambda(\frac{(K+1)^2}{2}-\frac{(K+1)}{2}-n(K+1)+\frac{n(n+1)}{2})}(-1)^{K+1-n}$$
$$=\frac{(q^\lambda;q^\lambda)_{m-1}q^{\lambda\frac{(K+1)(K+2)}{2}+\lambda\frac{n(n-1)}{2}-\lambda n(K+1)}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{m-K-2}(q^\lambda;q^\lambda)_{K+1-n}}$$
$$\times\frac{(q^s;q^\lambda)_{n+K+2}(q^s;q^\lambda)_{m+1}(-1)^{K+1-n}}{(q^s;q^\lambda)_{n+1}(q^s;q^\lambda)_{m+K+2}}$$
$$\times\frac{(1-q^{\lambda m+\lambda(K+1)^2+s(K+1)+\lambda(K+1)-s+r-\lambda n})}{(1-q^{\lambda m-\lambda n})(1-q^{\lambda(K+1)^2+s(K+1)+\lambda(K+1)-s+r})},$$

or, simplified,

$$-(1-q^{\lambda(K-n+1)})(1-q^{\lambda m+\lambda K^2+sK+\lambda K-s+r-\lambda n})(1-q^{r+2\lambda+3K\lambda+K^2\lambda+Ks})$$
$$\times(1-q^{\lambda(m+K+1)+s})+(1-q^{r-s+\lambda+2K\lambda-n\lambda+K^2\lambda+Ks})(1-q^{\lambda m-\lambda n})$$
$$\times(1-q^{r+\lambda+2K\lambda+m\lambda+K^2\lambda+Ks})(1-q^{2\lambda(K+1)+s})$$
$$=(1-q^{\lambda(m-K-1)})(1-q^{\lambda(n+K+1)+s})(1-q^{r-s+K\lambda+K^2\lambda+Ks})$$
$$\times(1-q^{r+2\lambda+3K\lambda+m\lambda-n\lambda+K^2\lambda+Ks})q^{\lambda+K\lambda-n\lambda},$$

which is a routine check. Thus we have the claimed result.

The other proofs could be done in a similar style, but are omitted here.

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