ON A RESULT OF BUNDER INVOLVING HORADAM SEQUENCES: A PROOF AND GENERALIZATION

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Abstract. This note introduces, proves, extends empirically and generalizes a short 1975 offering of M.W. Bunder who, in this journal, gave an isolated observation involving Horadam sequences on which work has been conducted for nearly half a century.

1. Introduction

Consider the sequence $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(w_0, w_1; p, q)\}_0^{\infty}$ defined, for given w_0, w_1, by the order two linear recurrence

$$w_n = pw_{n-1} - qw_{n-2}, \qquad n \geq 2, \tag{1.1}$$

particular values of p, q, w_0, w_1 giving rise to some well-known sequences (Fibonacci, Pell, Lucas, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Tagiuri, Fermat, Fermat-Lucas, for example). Such a general sequence is called a Horadam sequence, so named after the studies of A. F. Horadam begun in the 1960s and on which research has continued for almost half a century. For the most part it has been the case that the four defining parameters p, q, w_0, w_1 are real, although this does not apply in [1] where some new results on sequence periodicity in the complex plane have been established by the authors. Note that both types of Chebyshev polynomial—$T_n(x)$ (of the first kind) and $U_n(x)$ (of the second kind)—are solutions of (1.1) when $p = 2x$ and $q = 1$, with (for $n \geq 0$) $T_n(x) = w_n(1, x; 2x, 1)$ and $U_n(x) = w_n(1, 2x; 2x, 1)$.

Also recently, Larcombe et al. [3] have disseminated a literature survey on Horadam sequences in an attempt to identify and set down some of those various avenues of work conducted over the years; collectively, they display a diversity of ideas to give Horadam sequences their own profile within the broader field of linear recurrence theory whose literature is vast. A reference which has since come to light is due to M. W. Bunder [2], who in 1975 published in this journal a one-page note. While not in itself a deep result, his observation is nonetheless not without interest and permits a natural generalization.

2. Bunder's Result and its Generalization

Involving two initial values instances of the Horadam sequence $\{w_n(w_0, w_1; p, -q)\}_0^{\infty}$ (from the recurrence $w_n = pw_{n-1} + qw_{n-2}$), Bunder noted that, given $z_0 = a$, $z_1 = b$, the sequence (typeset with errors in [2])

$$\{z_n(a, b, p, q)\}_0^{\infty} = \{a, b, a^q b^p, a^{pq} b^{p^2+q}, a^{p^2q+q^2} b^{p^3+2pq}, a^{p^3q+2pq^2} b^{p^4+3p^2q+q^2}, \ldots\} \tag{2.1}$$

generated by the power product recurrence

$$z_n = (z_{n-1})^p (z_{n-2})^q, \qquad n \geq 2, \tag{2.2}$$

has, for $n \geq 0$, general $(n+1)$th term closed form

$$z_n = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)} \tag{2.3}$$

in terms of z_0, z_1 and the Horadam sequences seen. Its proof, omitted by Bunder, is a simple one by induction.

Proof. Defining sequences $\{t_n\}_0^\infty = \{w_n(1,0;p,-q)\}_0^\infty$ and $\{u_n\}_0^\infty = \{w_n(0,1;p,-q)\}_0^\infty$, we wish to show, given $z_0 = a$, $z_1 = b$, that $z_n = a^{t_n} b^{u_n}$ for $n \geq 0$.

Noting that the result holds for $n = 0, 1$ (since $z_0 = a = a^1 b^0 = a^{t_0} b^{u_0}$ and $z_1 = b = a^0 b^1 = a^{t_1} b^{u_1}$), we assume it is true for some $n = k, k-1$ $(k \geq 1)$. Consider, therefore, $z_{k+1} = (z_k)^p (z_{k-1})^q$ (by definition) $= (a^{t_k} b^{u_k})^p (a^{t_{k-1}} b^{u_{k-1}})^q$ (by hypothesis) $= a^{pt_k + qt_{k-1}} b^{pu_k + qu_{k-1}} = a^{t_{k+1}} b^{u_{k+1}}$ (by virtue of $\{t_n\}_0^\infty, \{u_n\}_0^\infty$ being second-order (Horadam) recurrence sequences as defined), with the inductive step upheld as required. Thus, the result is valid for $n \geq 0$. □

Given w_0, w_1, w_2, then denoting by $\{w_n(w_0, w_1, w_2; p, -q, -r)\}_0^\infty$ the sequence delivered by the extended Horadam type recurrence $w_n = pw_{n-1} + qw_{n-2} + rw_{n-3}$ $(n \geq 3)$, the next obvious (order three) case for consideration is

$$z_n = (z_{n-1})^p (z_{n-2})^q (z_{n-3})^r, \qquad n \geq 3, \tag{2.4}$$

for which, with initial values $z_0 = a$, $z_1 = b$, $z_2 = c$, we find

$$\{z_n(a,b,c,p,q,r)\}_0^\infty = \{a, b, c, a^r b^q c^p, a^{pr} b^{pq+r} c^{p^2+q}, a^{p^2 r + qr} b^{p^2 q + pr + q^2} c^{p^3 + 2pq + r}, \ldots\}, \tag{2.5}$$

with general term closed form

$$z_n = a^{w_n(1,0,0;p,-q,-r)} b^{w_n(0,1,0;p,-q,-r)} c^{w_n(0,0,1;p,-q,-r)}, \qquad n \geq 0, \tag{2.6}$$

where $\{w_n(1,0,0;p,-q,-r)\}_0^\infty = \{1, 0, 0, r, pr, p^2 r + qr, p^3 r + 2pqr + r^2, p^4 r + 3p^2 qr + 2pr^2 + q^2 r, \ldots\}$, $\{w_n(0,1,0;p,-q,-r)\}_0^\infty = \{0, 1, 0, q, pq + r, p^2 q + pr + q^2, p^3 q + p^2 r + 2pq^2 + 2qr, p^4 q + p^3 r + 3p^2 q^2 + 4pqr + q^3 + r^2, \ldots\}$ and $\{w_n(0,0,1;p,-q,-r)\}_0^\infty = \{0, 0, 1, p, p^2 + q, p^3 + 2pq + r, p^4 + 3p^2 q + 2pr + q^2, p^5 + 4p^3 q + 3p^2 r + 3pq^2 + 2qr, \ldots\}$. While the previous case is easily manageable by hand, this one is more demanding algebraically and has been verified computationally—a method most probably unavailable to Bunder in the 1970s as automated symbolic software was still in relative infancy. So, too, have the order four and five cases, and the general result is an obvious one as the pattern continues; we state this as a theorem.

Let $\{w_n(w_0, w_1, \ldots, w_{k-2}, w_{k-1}; h_1, -h_2, \ldots, -h_{k-1}, -h_k)\}_0^\infty$ be the sequence, with $k \geq 2$ starting values $w_0, w_1, \ldots, w_{k-2}, w_{k-1}$, arising from a kth order Horadam type linear recurrence

$$w_n = h_1 w_{n-1} + h_2 w_{n-2} + \cdots + h_{k-1} w_{n-k+1} + h_k w_{n-k}, \qquad n \geq k. \tag{2.7}$$

We write $w_n^{(p)}$ $(p = 1, \ldots, k)$ to denote the general term of that particular sequence for which all initial values are zero excepting $w_{p-1} = 1$, which is to say,

$$w_n^{(p)} = w_n(0, 0, \ldots, 0, 1 = w_{p-1}, 0, \ldots, 0, 0; h_1, -h_2, \ldots, -h_{k-1}, -h_k). \tag{2.8}$$

Theorem 2.1. *Given initial values* $z_0 = a_0$, $z_1 = a_1$, $\ldots$, $z_{k-2} = a_{k-2}$, $z_{k-1} = a_{k-1}$, *the general term*

$$z_n = z_n(a_0, a_1, \ldots, a_{k-2}, a_{k-1}, h_1, h_2, \ldots, h_{k-1}, h_k)$$

of the sequence generated by the multi-product recursion

$$z_n = \prod_{i=1}^{k} (z_{n-i})^{h_i} = (z_{n-1})^{h_1} (z_{n-2})^{h_2} \cdots (z_{n-k+1})^{h_{k-1}} (z_{n-k})^{h_k}, \qquad n \geq k,$$

has, for $n \geq 0$, a closed form

$$z_n = \prod_{p=0}^{k-1} (a_p)^{w_n^{(p+1)}} = (a_0)^{w_n^{(1)}} (a_1)^{w_n^{(2)}} \cdots (a_{k-2})^{w_n^{(k-1)}} (a_{k-1})^{w_n^{(k)}}.$$

We do not give a proof (which is left as an exercise for any interested reader) since it is merely an extended version of the order two case proof, running along the same inductive line of argument.

3. Summary

This short note adds to the body of knowledge assimilated in the aforementioned survey on Horadam sequences [3]. It would appear that since Bunder's 1975 publication [2] no mention has been made of his observation nor the natural generalization given here. On the order two case result itself, we remark that a first principles constructive approach to its proof would offer more insight than an inductive one, and this will be the subject of further discussion elsewhere.

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References

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