ESTIMATING THE APÉRY NUMBERS II

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Abstract. We describe a method to find the complete asymptotic expansion of the nth Apéry number, and find the first few terms.

1. Introduction

In 1983, Apéry stunned the mathematical world by proving that $\zeta(3)$ is irrational. Alf van der Poorten gave an entertaining account of Apéry's presentation [3].

Apéry's proof involves the eponymous numbers,

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

that satisfy the recurrence

$$(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0$$

together with $A_0 = 1,\ A_1 = 5$.

In a previous note [1] I demonstrated how one can find the dominant term in the asymptotic expansion of A_n. Indeed, I found that

$$A_n \sim Cn^{-\frac{3}{2}}\alpha^n \quad \text{as } n \to \infty,$$

where

$$C = \frac{3+2\sqrt{2}}{2^{\frac{9}{4}}\pi^{\frac{3}{2}}} \quad \text{and} \quad \alpha = 17 + 12\sqrt{2}.$$

In this note I show how one can use the recurrence given above to find as many terms of the asymptotic expansion of A_n as one might want.

Indeed, I show that

$$A_n \sim Cn^{-\frac{3}{2}}\alpha^n \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} \ \cdots \right) \quad \text{as } n \to \infty$$

where

$$c_1 = \frac{15\sqrt{2}-48}{64}, \quad c_2 = \frac{2057-1200\sqrt{2}}{4096}, \quad c_3 = \frac{62917\sqrt{2}-87024}{262144},$$

and so on.

Note that the same method can be used to find the asymptotic expansion of a large class of recurrent sequences, even if one doesn't have the solution in closed form!

2. The calculation

We have the recurrence

$$(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)A_n + n^3 A_{n-1} = 0.$$

If we divide by n^3, we find

$$\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)A_{n+1}-\left(34+\frac{51}{n}+\frac{27}{n^2}+\frac{5}{n^3}\right)A_n+A_{n-1}=0.$$

Now suppose

$$A_n=Cn^{-\frac{3}{2}}\alpha^n\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\ \cdots\right).$$

Then

$$\begin{aligned}A_{n+1}&=C(n+1)^{-\frac{3}{2}}\alpha^{n+1}\left(1+\frac{c_1}{n+1}+\frac{c_2}{(n+1)^2}+\ \cdots\right)\\&=Cn^{-\frac{3}{2}}\left(1+\frac{1}{n}\right)^{-\frac{3}{2}}\alpha^{n+1}\left(1+\frac{c_1}{n}\left(1+\frac{1}{n}\right)^{-1}+\frac{c_2}{n^2}\left(1+\frac{1}{n}\right)^{-2}+\ \cdots\right)\\&=Cn^{-\frac{3}{2}}\alpha^{n+1}\left(1-\frac{3}{2n}+\frac{15}{8n^2}-\frac{35}{16n^3}+\ \cdots\right)\\&\quad\times\left(1+\frac{c_1}{n}+\frac{c_2-c_1}{n^2}+\frac{c_3-2c_2+c_1}{n^3}+\ \cdots\right),\end{aligned}$$

while

$$\begin{aligned}A_{n-1}&=C(n-1)^{-\frac{3}{2}}\alpha^{n-1}\left(1+\frac{c_1}{n-1}+\frac{c_2}{(n-1)^2}+\ \cdots\right)\\&=Cn^{-\frac{3}{2}}\left(1-\frac{1}{n}\right)^{-\frac{3}{2}}\alpha^{n-1}\left(1+\frac{c_1}{n}\left(1-\frac{1}{n}\right)^{-1}+\frac{c_2}{n^2}\left(1-\frac{1}{n}\right)^{-2}+\ \cdots\right)\\&=Cn^{-\frac{3}{2}}\alpha^{n-1}\left(1+\frac{3}{2n}+\frac{15}{8n^2}+\frac{35}{16n^3}+\ \cdots\right)\\&\quad\times\left(1+\frac{c_1}{n}+\frac{c_2+c_1}{n^2}+\frac{c_3+2c_2+c_1}{n^3}+\ \cdots\right).\end{aligned}$$

If we substitute these results into the recurrence relation and divide by $Cn^{-\frac{3}{2}}\alpha^n$, we obtain

$$\begin{aligned}&\alpha\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)\left(1-\frac{3}{2n}+\frac{15}{8n^2}-\frac{35}{16n^3}+\ \cdots\right)\\&\quad\times\left(1+\frac{c_1}{n}+\frac{c_2-c_1}{n^2}+\frac{c_3-2c_2+c_1}{n^3}+\ \cdots\right)\\&\quad-\left(34+\frac{51}{n}+\frac{27}{n^2}+\frac{5}{n^3}\right)\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\ \cdots\right)\\&\quad+\alpha^{-1}\left(1+\frac{3}{2n}+\frac{15}{8n^2}+\frac{35}{16n^3}+\ \cdots\right)\left(1+\frac{c_1}{n}+\frac{c_2+c_1}{n^2}+\frac{c_3+2c_2+c_1}{n^3}+\cdots\right)\\&\quad=0,\end{aligned}$$

or,

$$\alpha\left(1+\frac{3}{2n}+\frac{3}{8n^2}-\frac{1}{16n^3}+\ \cdots\right)\left(1+\frac{c_1}{n}+\frac{c_2-c_1}{n^2}+\frac{c_3-2c_2+c_1}{n^3}+\ \cdots\right)$$
$$-\left(34+\frac{51}{n}+\frac{27}{n^2}+\frac{5}{n^3}\right)\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\ \cdots\right)$$
$$+\alpha^{-1}\left(1+\frac{3}{2n}+\frac{15}{8n^2}+\frac{35}{16n^3}+\ \cdots\right)\left(1+\frac{c_1}{n}+\frac{c_2+c_1}{n^2}+\frac{c_3+2c_2+c_1}{n^3}+\cdots\right)$$
$$=0.$$

If we now compare coefficients, we obtain the equations

$$\alpha-34+\alpha^{-1}=0,$$
$$\alpha\left(\frac{3}{2}+c_1\right)-(51+34c_1)+\alpha^{-1}\left(\frac{3}{2}+c_1\right)=0,$$
$$\alpha\left(\frac{3}{8}+\frac{3}{2}c_1+(c_2-c_1)\right)-(27+51c_1+34c_2)+\alpha^{-1}\left(\frac{15}{8}+\frac{3}{2}c_1+(c_2+c_1)\right)=0,$$
$$\alpha\left(-\frac{1}{16}+\frac{3}{8}c_1+\frac{3}{2}(c_2-c_1)+(c_3-2c_2+c_1)\right)-(5+27c_1+51c_2+34c_3)$$
$$+\alpha^{-1}\left(\frac{35}{16}+\frac{15}{8}c_1+\frac{3}{2}(c_2+c_1)+(c_3+2c_2+c_1)\right)=0,$$
$$\alpha\left(\frac{3}{128}-\frac{1}{16}c_1+\frac{3}{8}(c_2-c_1)+\frac{3}{2}(c_3-2c_2+c_1)+(c_4-3c_3+3c_2-c_1)\right)$$
$$-(5c_1+27c_2+51c_3+34c_4)$$
$$+\alpha^{-1}\left(\frac{315}{128}+\frac{35}{16}c_1+\frac{15}{8}(c_2+c_1)+\frac{3}{2}(c_3+2c_2+c_1)+(c_4+3c_3+3c_2+c_1)\right)$$
$$=0,$$

and so on.

The solutions of these are

$$c_1=\frac{15\sqrt{2}-48}{64},\quad c_2=\frac{2057-1200\sqrt{2}}{4096},\quad c_3=\frac{62917\sqrt{2}-87024}{262144},$$

and so on. Thus we have the result stated earlier.

Note added: Shalosh B. Ekhad and Doron Zeilberger [2] have automated this whole process (apart from the exact determination of C).

References

[1] M. D. Hirschhorn, *Estimating the Apéry numbers*, The Fibonacci Quarterly, **50.2** (2012), 129–131.

[2] S. B. Ekhad and D. Zeilberger, *AsyRec: A Maple package for computing asymptotics of solutions of linear recurrence, equations with polynomial coefficients*, April, 2008, `http://www.math.rutgers.edu/~zeilberg/pj.html`

[3] A. van der Poorten, *A proof the Euler missed ...*, The Mathematical Intelligencer, **1.4** (1979), 195–203.

MSC2010: 41A60

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