A NAIVE PROOF THAT $F_{5n} \equiv 0 \pmod 5$

MICHAEL D. HIRSCHHORN

Abstract. We give a new and simple proof of the fact that

$$F_{5n} \equiv 0 \pmod 5$$

and more.

1. Introduction

We give a new and simple proof of the fact that, modulo 5

$$F_{5n} \equiv 0,$$

as well as the facts that

$$F_{5n+1} \equiv F_{5n+2} \equiv F_{n+1} + 2F_n = F_{n+2} + F_n,$$
$$F_{5n+3} \equiv -F_{5n+4} \equiv 2F_{n+1} - F_n = F_{n+1} + F_{n-1}.$$

2. Proofs

We have

$$\begin{aligned}\sum_{n\geq 0} F_n x^n &= \frac{x}{1-x-x^2} \\ &= \frac{x(1-x-x^2)^4}{(1-x-x^2)^5} \\ &= \frac{x(1-4x+2x^2+8x^3-5x^4-8x^5+2x^6+4x^7+x^8)}{1-5x+5x^2+10x^3-15x^4-11x^5+15x^6+10x^7-5x^8-5x^9-x^{10})} \\ &\equiv \frac{x+x^2+2x^3-2x^4+2x^6+2x^7-x^8+x^9}{1-x^5-x^{10}} \pmod 5.\end{aligned}$$

It follows that, modulo 5,

$$\sum_{n\geq 0} F_{5n+1} x^n \equiv \frac{1+2x}{1-x-x^2},$$

$$\sum_{n\geq 0} F_{5n+2} x^n \equiv \frac{1+2x}{1-x-x^2},$$

$$\sum_{n\geq 0} F_{5n+3} x^n \equiv \frac{2-x}{1-x-x^2},$$

$$\sum_{n\geq 0} F_{5n+4} x^n \equiv \frac{-2+x}{1-x-x^2}$$

$$\text{and} \quad \sum_{n\geq 0} F_{5n}x^n \equiv 0.$$

It follows that, modulo 5,

$$F_{5n} \equiv 0 \tag{2.1}$$

and

$$F_{5n+1} \equiv F_{5n+2} \equiv F_{n+1} + 2F_n = F_{n+2} + F_n, \tag{2.2}$$

$$F_{5n+3} \equiv -F_{5n+4} \equiv 2F_{n+1} - F_n = F_{n+1} + F_{n-1}. \tag{2.3}$$

3. Comments

In the past, I have proved that $F_{5n} \equiv 0 \pmod 5$ by finding the generating function. The method involves a fifth root of unity, η.

Thus, we start by writing

$$\frac{x}{1-x-x^2}$$
$$= \frac{x(1-\eta x-\eta^2x^2)(1-\eta^2x-\eta^4x^2)(1-\eta^3x-\eta^6x^2)(1-\eta^4x-\eta^8x^2)}{(1-x-x^2)(1-\eta x-\eta^2x^2)(1-\eta^2x-\eta^4x^2)(1-\eta^3x-\eta^6x^2)(1-\eta^4x-\eta^8x^2)}.$$

The idea is that the denominator is now a function of x^5. For if we write $D(x)$ for the denominator, then

$$D(\eta x) = D(x).$$

If we write

$$D(x) = \sum_{n\geq 0} d_n x^n$$

it follows that

$$\eta^n d_n = d_n,$$

so

$$d_n = 0$$

whenever $5 \nmid n$.

Indeed, using the facts that

$$\eta^5 = 1 \quad \text{and} \quad 1+\eta+\eta^2+\eta^3+\eta^4 = 0,$$

it is not too hard to show that the above equation becomes

$$\frac{x}{1-x-x^2} = \frac{x(1+x+2x^2+3x^3+5x^4-3x^5+2x^6-x^7+x^8)}{1-11x^5-x^{10}}.$$

Of course, this can be checked by cross–multiplication. Indeed, it can be stated without derivation, and then verified. In any case, we obtain

$$\sum_{n\geq 0} F_{5n+1}x^n = \frac{1-3x}{1-11x-x^2},$$

$$\sum_{n\geq 0} F_{5n+2}x^n = \frac{1+2x}{1-11x-x^2},$$

$$\sum_{n\geq 0} F_{5n+3}x^n = \frac{2-x}{1-11x-x^2},$$

$$\sum_{n\geq 0} F_{5n+4}x^n = \frac{3+x}{1-11x-x^2}$$

$$\text{and} \quad \sum_{n\geq 0} F_{5n}x^n = \frac{5x}{1-11x-x^2}.$$

In particular, it follows that

$$F_{5n} \equiv 0 \pmod 5.$$

The new proof presented in this paper is more naive, in that it does not require reference to roots of unity.

MSC2010: 11B39

School of Mathematics and Statistics, UNSW, Sydney, Australia 2052
E-mail address: m.hirschhorn@unsw.edu.au