NEW VIÈTE-LIKE INFINITE PRODUCTS OF NESTED RADICALS WITH FIBONACCI AND LUCAS NUMBERS

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Abstract. In a 2007 contribution by Osler in this Quarterly, the so-named Vieta-like products were introduced as two eye-catching formulas representing either the nth Fibonacci number in terms of a product of nested radicals with the nth Lucas number inside, or vice-versa. As the original and famous Viète's infinite product, Osler's infinite products have plus signs inside the nested radicals. In this paper we explore infinite products of nested square roots with Fibonacci and Lucas numbers with the novelty that inside the radical symbols there are minus signs instead of plus signs.

1. Introduction

In a recent contribution [5], Osler gave two striking Viète-like infinite products, which can be rewritten as

$$\frac{a_n}{2n\ln\phi} = \sqrt{\frac{1}{2}+\frac{b_n}{4}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{b_n}{4}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{b_n}{4}}}}\cdots, \tag{1.1}$$

where

$$a_n = \begin{cases} \sqrt{5}F_n & \text{if } n \text{ is even,} \\ L_n & \text{if } n \text{ is odd,} \end{cases} \qquad b_n = \begin{cases} L_n & \text{if } n \text{ is even,} \\ \sqrt{5}F_n & \text{if } n \text{ is odd.} \end{cases}$$

As usual, F_n and L_n, respectively stand for the nth Fibonacci and Lucas numbers, and ϕ denotes the golden ratio. To derive (1.1), Osler first used a standard procedure to deduce Viète-like infinite products, using the hyperbolic sine and cosine functions instead of the trigonometric functions, and next he evaluated $\sinh x$ and $\cosh x$ for the values $x = n\log\phi$, n being a positive integer. (As Osler said, it was Richard Askey who showed him how the Fibonacci and Lucas numbers were related to the hyperbolic functions.)

In this paper we give new infinite products of nested square roots which link the nth Fibonacci and Lucas numbers to each other, and which have some resemblances with the infinite products introduced by Osler in [5]. Concretely, we will prove that

$$\frac{i\sqrt{5}F_n}{\sqrt{3}} = \sqrt{2-L_n}\sqrt{2-\sqrt{2-L_n}}\sqrt{2-\sqrt{2-\sqrt{2-L_n}}}\cdots, \qquad n \text{ even,} \tag{1.2}$$

and also its twin counterpart

$$\frac{iL_n}{\sqrt{3}} = \sqrt{2-\sqrt{5}F_n}\sqrt{2-\sqrt{2-\sqrt{5}F_n}}\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{5}F_n}}}\cdots, \qquad n \text{ odd.} \tag{1.3}$$

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Here and subsequently, i stands for the imaginary unit, and the symbol $\sqrt{\cdot}$ will stand for the principal value of the complex square root function. Thus, for a nonnegative real x, $\sqrt{x}$ stands for its nonnegative square root, and for a nonzero complex number z, $\sqrt{z} = \sqrt{|z|}e^{i\frac{\mathrm{Arg}(z)}{2}}$, where $|\cdot|$ is the modulus function and $\mathrm{Arg}(\cdot)$ is the principal value of the argument function, which takes values in the interval $(-\pi, \pi]$.

2. Two Key Tools Towards the Main Result

In this section our purpose is to derive a Viète-like infinite product with just minus signs inside the nested radicals. To achieve this aim we first give an infinite product of cosines (first key tool), which essentially coincides with [2, Theorem 1], but here in the complex setting and with a shorter proof. Next we state the main result in [3] (second key tool), which connects certain values of the cosine function with certain nested radicals, and which generalizes to the whole complex plane previous results by L. D. Servi [7], for the interval $[-1, 1]$, and by M. A. Nyblom [4], for $[1, \infty)$. Finally, we show how these two results lead us to the desired Viète-like infinite product.

Proposition 2.1. *(cf.* [2, Theorem 1, p.16]*) Let $a_j = (2^j - (-1)^j)/3$ denote the jth Jacobsthal number, recursively defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$* [8]. *For any complex z,*

$$\frac{2}{\sqrt{3}} \cos z = \prod_{n=0}^{\infty} 2 \cos \left(\frac{a_{n+2}\pi}{2^{n+2}} + \frac{(-1)^n z}{2^{n+1}} \right). \tag{2.1}$$

Proof. Our first step consists of proving that for each positive integer k,

$$\cos z = \left(\prod_{j=0}^{k-1} 2 \cos \left(\frac{a_{j+2}\pi}{2^{j+2}} + \frac{(-1)^j z}{2^{j+1}} \right) \right) \cos \left(\frac{a_k \pi}{2^{k+1}} + \frac{(-1)^k z}{2^k} \right), \qquad z \in \mathbb{C}. \tag{2.2}$$

The proof goes by induction on k.

First, the base case: we need to verify that (2.2) holds for $k = 1$. To this end, and taking into account that $a_2 = 1$, we must verify that

$$\cos z = 2 \cos \left(\frac{\pi}{4} + \frac{z}{2} \right) \cos \left(\frac{\pi}{4} - \frac{z}{2} \right), \tag{2.3}$$

which trivially holds.

Second, the induction step: assuming our claim to hold for $k = p$, we will prove it for $p+1$. Rewriting equation (2.2) with $k = p$, and applying (2.3) we get

$$\begin{aligned} \cos z &= \left(\prod_{j=0}^{p-1} 2 \cos \left(\frac{a_{j+2}\pi}{2^{j+2}} + \frac{(-1)^j z}{2^{j+1}} \right) \right) 2 \cos \left(\frac{(2^p + a_p)\pi}{2^{p+2}} + \frac{(-1)^p z}{2^{p+1}} \right) \\ &\quad \times \cos \left(\frac{(2^p - a_p)\pi}{2^{p+2}} + \frac{(-1)^{p+1} z}{2^{p+1}} \right) \\ &= \left(\prod_{j=0}^{p} 2 \cos \left(\frac{a_{j+2}\pi}{2^{j+2}} + \frac{(-1)^j z}{2^{j+1}} \right) \right) \cos \left(\frac{a_{p+1}\pi}{2^{p+2}} + \frac{(-1)^{p+1} z}{2^{p+1}} \right), \end{aligned}$$

and (2.2) is proved.

Our next concern will be to deduce (2.1) from (2.2).

Since

$$\lim_{k\to\infty} \cos\left(\frac{a_k\pi}{2^{k+1}} + \frac{(-1)^k x}{2^k}\right) = \cos\left(\lim_{k\to\infty}\left(\frac{a_k\pi}{2^{k+1}} + \frac{(-1)^k x}{2^k}\right)\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

then we conclude

$$\frac{2}{\sqrt{3}}\cos z = \lim_{k\to\infty} \frac{1}{\cos\left(\frac{a_k\pi}{2^{k+1}} + \frac{(-1)^k x}{2^k}\right)} \lim_{k\to\infty}\cos z = \lim_{k\to\infty}\prod_{j=0}^{k-1} 2\cos\left(\frac{a_{j+2}\pi}{2^{j+2}} + \frac{(-1)^j z}{2^{j+1}}\right),$$

which completes the proof. □

We can now proceed to rewrite Proposition 2.1 as a Viète-like formula. For this to be done we must link each cosine in formula (2.1) with a certain nested radical. And this can be achieved by using the main result in [3], which we repeat here without proof, thus making our exposition self-contained.

Proposition 2.2. [3, Theorem 4, p. 71] *Let ξ be an arbitrary complex number and let k be a positive integer. If $b_l \in \{1, -1\}$ for $l = 1, 2, \ldots, k$, then*

$$\frac{b_k}{2}\sqrt{2 + b_{k-1}\sqrt{2 + b_{k-2}\sqrt{2 + \cdots + b_1\sqrt{2 + 2\xi}}}} = \cos\left(\left(2^{-1} - \sum_{j=1}^{k+1}(2^{-(j+1)}\prod_{i=0}^{j-1} b_{k-i})\right)\pi\right)$$

$$= \cos\left(\left(\frac{1}{2} - \frac{b_k}{2^2} - \frac{b_k b_{k-1}}{2^3} - \cdots - \frac{b_k b_{k-1}\cdots b_1 b_0}{2^{k+2}}\right)\pi\right), \tag{2.4}$$

where

$$b_0 = \frac{4}{\pi}\mathrm{Arcsin}\,\xi = \frac{4}{\pi}\mathrm{Arg}\big(i\xi + \sqrt{1-\xi^2}\big) - i\frac{4}{\pi}\ln|i\xi + \sqrt{1-\xi^2}|.$$

We are thus led to the following strengthening of [2, Theorem 3, p.18].

Proposition 2.3. *For any $w \in \mathbb{C}$,*

$$\sqrt{\frac{4-w^2}{3}} = \sqrt{2-w}\sqrt{2-\sqrt{2-w}}\sqrt{2-\sqrt{2-\sqrt{2-w}}}\cdots. \tag{2.5}$$

Proof. For a fixed complex number w and for a nonnegative integer n let us introduce the coefficients $b_j = b_j(n)$, defined by means of

$$b_0 = -\frac{4}{\pi}\mathrm{Arcsin}\frac{w}{2}, \qquad b_1 = b_2 = \cdots = b_n = -1, \qquad b_{n+1} = 1,$$

in case that $n > 0$, and

$$b_0 = -\frac{4}{\pi}\mathrm{Arcsin}\frac{w}{2}, \qquad b_1 = 1,$$

when $n = 0$. Then, formula (2.4) transforms to

$$b_{n+1}\sqrt{2 + b_n\sqrt{2 + \cdots + b_1\sqrt{2 + 2\sin b_0\pi/4}}} = \underbrace{\sqrt{2-\sqrt{2-\cdots-\sqrt{2-w}}}}_{n+1 \text{ square roots}}$$

$$= 2\cos\left(\left(\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \cdots + (-1)^{n+1}\frac{1}{2^{n+2}} + (-1)^{n+2}\frac{4\mathrm{Arcsin}(w/2)}{2^{n+3}\pi}\right)\pi\right). \tag{2.6}$$

Evaluating the above geometric series, we have

$$\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\cdots+(-1)^{n+1}\frac{1}{2^{n+2}}=\frac{2^{n+2}-(-1)^{n+2}}{3\cdot 2^{n+2}}=\frac{a_{n+2}}{2^{n+2}}.$$

Thus, if we define $z=\mathrm{Arcsin}(w/2)$, equation (2.6) transforms to

$$\underbrace{\sqrt{2-\sqrt{2-\cdots-\sqrt{2-w}}}}_{n+1\text{ square roots}}=2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}}+\frac{(-1)^n z}{2^{n+1}}\right).$$

Substituting the equation above in (2.1), and taking into account that $\cos z=\cos(\mathrm{Arcsin}(w/2))$ $=\sqrt{1-(w^2/4)}$, we establish (2.5) as desired. □

Proposition 2.3 is not new. To the best of our knowledge, it must be credited to A. Levin [1, formula (48)], who gave a different proof. Our proof mimics the one in [2, Theorem 3, p.18], but uses Proposition 2.2 to generalize the former result to the whole complex plane.

3. Main Result: A Mirror Image of Osler's Formulas

A well-known quadratic relation between Fibonacci and Lucas numbers is [6, p. 5]

$$L_n^2-5F_n^2=(-1)^n 4,\qquad n=1,2,3,\ldots. \tag{3.1}$$

Clearly (3.1) can be rewritten as

$$i\sqrt{5}F_n=\sqrt{4-L_n^2},\qquad \text{for even } n, \tag{3.2}$$

$$iL_n=\sqrt{4-5F_n^2},\qquad \text{for odd } n. \tag{3.3}$$

Having disposed of this preliminary step, we can now proceed to state our target formulas (1.2) and (1.3).

Theorem 3.1. *Let n be a positive integer. We have*

$$\frac{ia_n}{\sqrt{3}}=\sqrt{2-b_n}\sqrt{2-\sqrt{2-b_n}}\sqrt{2-\sqrt{2-\sqrt{2-b_n}}}\cdots, \tag{3.4}$$

where

$$a_n=\begin{cases}\sqrt{5}F_n & \text{if } n \text{ is even,}\\ L_n & \text{if } n \text{ is odd,}\end{cases}\qquad b_n=\begin{cases}L_n & \text{if } n \text{ is even,}\\ \sqrt{5}F_n & \text{if } n \text{ is odd.}\end{cases}$$

Proof. The proof falls naturally into two parts.

First fix n a positive even integer. Replacing w by L_n in (2.5), and taking into account (3.2) to transform the right-hand side of the equality, we get (3.4) for even n.

A similar method works when n is a positive odd integer. □

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