THE GOLDEN SEQUENCE

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Abstract. This paper considers the sequence of fractional parts of multiples of the golden ratio. The main result characterizes the Fibonacci numbers by minimizing or maximizing this sequence.

1. Introduction and Preliminaries

For the Golden Section, there are two ratios—ratio 'big/small' and reciprocal ratio 'small/big'—which are calculated as follows

$$\Phi = \frac{\sqrt{5}+1}{2} = 1.618\ldots \text{ and } \psi = \frac{\sqrt{5}-1}{2} = 0.618\ldots.$$

Obviously,

$$\Phi + \psi = \sqrt{5} \text{ and } \psi^2 = 1 - \psi.$$

The Fibonacci numbers are defined recursively as $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \qquad (n = 1, 2, 3, \ldots).$$

In this paper, i, k, n always denote integers greater than or equal to 1. Most properties of Φ, ψ, and the Fibonacci sequence can be found in the well-known reference work [4]. In particular, Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\big(\Phi^n - (-\psi)^n\big). \tag{1.1}$$

This formula allows both

- direct calculation of F_n from Φ and
- calculating the infinitesimal difference between $\frac{F_{n+1}}{F_n}$ and Φ.

The latter means that $\frac{F_{n+1}}{F_n} - \Phi = \frac{(-\psi)^n}{F_n}$ and derives from:

$$F_{n+1} - \Phi F_n \overset{(1.1)}{=} \frac{1}{\sqrt{5}}\big(-(-\psi)^{n+1} + \Phi(-\psi)^n\big) = (-\psi)^n \frac{1}{\sqrt{5}}(\psi + \Phi) = (-\psi)^n.$$

Moreover, this proves $F_{n+1} = \Phi F_n + (-\psi)^n$ and, hence,

$$\Phi F_n + (-\psi)^n \text{ is a positive integer.} \tag{1.2}$$

For real x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\lceil x \rceil$ denote the least integer greater than or equal to x. Let $\langle x \rangle = x - \lfloor x \rfloor$. Note that for real x, y, and δ,

$$\langle x + y \rangle = \begin{cases} \langle x \rangle + \langle y \rangle - 1; & \text{if } \langle x \rangle + \langle y \rangle \geq 1, \\ \langle x \rangle + \langle y \rangle; & \text{if } \langle x \rangle + \langle y \rangle < 1, \end{cases} \tag{1.3}$$

and

$$x + \delta \text{ is a positive integer and } -1 < \delta < 1 \text{ than } x + \delta = \begin{cases} \lceil x \rceil; & \text{if } \delta > 0, \\ \lfloor x \rfloor; & \text{if } \delta \leq 0. \end{cases} \quad (1.4)$$

These formulas will be useful later.

2. The Golden Sequence

The Golden Sequence is defined by the fractional parts of $n\Phi$, that is

$$\langle \Phi \rangle, \langle 2\Phi \rangle, \langle 3\Phi \rangle, \dots .$$

Since $\Phi = 1 + \psi$, it holds $\langle n\Phi \rangle = \langle n\psi \rangle$ for all $n \geq 1$. Figure 1 illustrates the initial values of the Golden Sequence, using different symbols for $\langle n\Phi \rangle$ if n is a Fibonacci number.

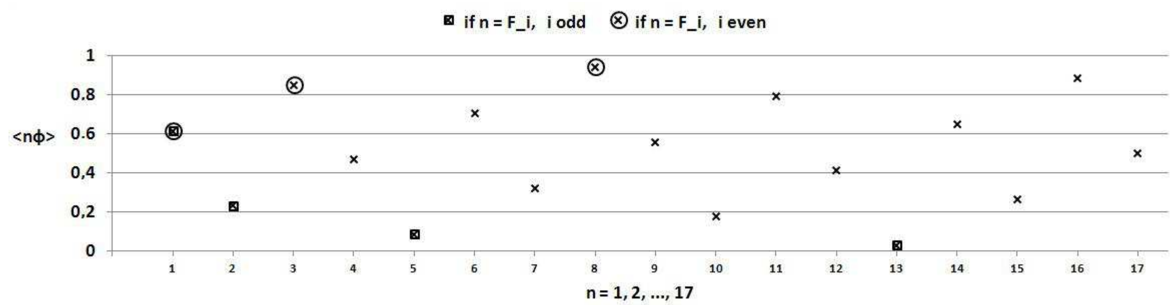

Figure 1. The Golden Sequence – Fibonacci elements emphasized.

The subsequences $\langle F_1\Phi \rangle, \langle F_3\Phi \rangle, \langle F_5\Phi \rangle, \langle F_7\Phi \rangle, \dots$ and $\langle F_2\Phi \rangle, \langle F_4\Phi \rangle, \langle F_6\Phi \rangle, \dots$ are monotone. This results from the following lemma.

Lemma 2.1. *(cf.* [3], *p. 85, exercise 31). For all integers* $k \geq 1$

$$\langle F_{2k-1}\Phi \rangle = \psi^{2k-1} \text{ and } \langle F_{2k}\Phi \rangle = 1 - \psi^{2k}.$$

Proof. By (1.2) and (1.4), $F_{2k-1}\Phi + (-\psi)^{2k-1} = \lfloor F_{2k-1}\Phi \rfloor$. Thus, $\langle F_{2k-1}\Phi \rangle = \psi^{2k-1}$. For the second part, again by (1.2) and (1.4), $F_{2k}\Phi + (-\psi)^{2k} = \lceil F_{2k}\Phi \rceil = \lfloor F_{2k}\Phi \rfloor + 1$. Thus, $\langle F_{2k}\Phi \rangle = 1 - \psi^{2k}$. □

Now, the Fibonacci elements—odd or even subscripts—of the Golden Sequence are shown to be extreme—minimal or maximal, respectively—until the next Fibonacci element but one.

Lemma 2.2. *For all integers* $k \geq 1$
(*a*) $\psi^{2k-1} \leq \langle i\Phi \rangle$ *for all* $1 \leq i < F_{2k+1}$ *and*
(*b*) $1 - \psi^{2k} \geq \langle i\Phi \rangle$ *for all* $1 \leq i < F_{2k+2}$.

Proof. By 'interlaced' induction on k, (a) holds for $k = 1$ since $F_3 = 2$ and $\psi^1 \leq \langle \Phi \rangle$. ($b$) holds for $k = 1$ since $F_4 = 3$ and $1 - \psi^2 = \psi \geq \langle \Phi \rangle$, $1 - \psi^2 = \psi \geq 2\psi - 1 = \langle 2\psi \rangle = \langle 2\Phi \rangle$.

Suppose (a) and (b) hold for k.

(a) For $k + 1$, (a) follows since

$$\psi^{2k+1} < \langle i\Phi \rangle \text{ for all } F_{2k+1} < i < F_{2k+3}. \quad (2.1)$$

For arbitrary i_0 between F_{2k+1} and F_{2k+3} there exists $j_0 < F_{2k+2}$ such that $i_0 = F_{2k+1} + j_0$.

By (b), $1-\psi^{2k} \geq \langle j_0\Phi\rangle$. By Lemma 2.1, $\langle F_{2k+1}\Phi\rangle = \psi^{2k+1}$. Thus, $\langle F_{2k+1}\Phi\rangle + \langle j_0\Phi\rangle \leq \psi^{2k+1} + 1 - \psi^{2k} < 1$. Equation (1.3) yields $\langle i_0\Phi\rangle = \langle F_{2k+1}\Phi + j_0\Phi\rangle = \langle F_{2k+1}\Phi\rangle + \langle j_0\Phi\rangle$. As $\langle j_0\Phi\rangle > 0$ it follows $\langle i_0\Phi\rangle > \langle F_{2k+1}\Phi\rangle = \psi^{2k+1}$ and (2.1) is proved.

(b) For $k+1$, (b) follows since

$$1-\psi^{2k+2} > \langle i\Phi\rangle \text{ for all } F_{2k+2} < i < F_{2k+4}. \tag{2.2}$$

For arbitrary i_0 between F_{2k+2} and F_{2k+4} there exists $j_0 < F_{2k+3}$ such that $i_0 = F_{2k+2} + j_0$.

By (a) for $k+1$, $\psi^{2k+1} \leq \langle j_0\Phi\rangle$. Again by Lemma 2.1, $\langle F_{2k+2}\Phi\rangle = 1-\psi^{2k+2}$. Thus, $\langle F_{2k+2}\Phi\rangle + \langle j_0\Phi\rangle \geq 1-\psi^{2k+2}+\psi^{2k+1} > 1$. Equation (1.3) yields $\langle i_0\Phi\rangle = \langle F_{2k+2}\Phi + j_0\Phi\rangle = \langle F_{2k+2}\Phi\rangle + \langle j_0\Phi\rangle - 1$. As $\langle j_0\Phi\rangle - 1 < 0$ it follows $\langle i_0\Phi\rangle < \langle F_{2k+2}\Phi\rangle = 1-\psi^{2k+2}$ and (2.2) is proved. □

Theorem 2.3. *A positive integer n is a Fibonacci number if and only if*

(a) $\langle n\Phi\rangle < \langle i\Phi\rangle$ *for all* $1 \leq i < n$ *or*
(b) $\langle n\Phi\rangle > \langle i\Phi\rangle$ *for all* $1 \leq i < n$.

Moreover,

(a) *holds if and only if* $n = F_{2k-1}$ *for some* $k \geq 1$, *and*
(b) *holds if and only if* $n = F_{2k}$ *for some* $k \geq 1$.

Proof. It suffices to show the 'moreover' parts. Assume (a) holds. Let k be maximal such that $F_{2k-1} \leq n$. Thus, $F_{2k+1} > n$. By Lemma 2.2, it follows that $\psi^{2k-1} \leq \langle n\Phi\rangle$. Now, $F_{2k-1} = n$ will be shown by *reductio ad absurdum*. Suppose $F_{2k-1} < n$. By (a), $\langle n\Phi\rangle < \langle F_{2k-1}\Phi\rangle = \psi^{2k-1}$. This contradicts $\psi^{2k-1} \leq \langle n\Phi\rangle$.

Conversely, let $n = F_{2k-1}$ for some $k \geq 1$. (a) is true if $k=1$, since $n = F_1 = 1$. If $k > 1$, by Lemma 2.2, for all $1 \leq i < F_{2k-1}$, $\langle i\Phi\rangle \geq \psi^{2k-3} > \psi^{2k-1} = \langle F_{2k-1}\Phi\rangle = \langle n\Phi\rangle$.

Suppose (b) holds. Let k be maximal such that $F_{2k} \leq n$. Thus, $F_{2k+2} > n$. By Lemma 2.2, it follows $1-\psi^{2k} \geq \langle n\Phi\rangle$. Again, $F_{2k} = n$ will be shown by *reductio ad absurdum*. Suppose $F_{2k} < n$. By (b), $\langle n\Phi\rangle > \langle F_{2k}\Phi\rangle = 1-\psi^{2k}$. This contradicts $1-\psi^{2k} \geq \langle n\Phi\rangle$.

Conversely, let $n = F_{2k}$ for some $k \geq 1$. (b) is true if $k=1$, since $n = F_2 = 1$. If $k > 1$, by Lemma 2.2, for all $1 \leq i < F_{2k}$ $\langle i\Phi\rangle \leq 1-\psi^{2k-2} < 1-\psi^{2k} = \langle F_{2k}\Phi\rangle = \langle n\Phi\rangle$. □

Characterizing conditions, other than that of Theorem 2.3, are stated in [1], i.e., a positive integer n is a Fibonacci number if and only if $5n^2-4$ or $5n^2+4$ is a complete square. This is a special case of a more general result from [2].

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