ON SUMS OF PRODUCTS OF FIBONACCI-TYPE RECURRENCES

M. A. KHAN AND HARRIS KWONG

Abstract. We derive two formulas for the summation $\sum_{i=0}^{n} C_{r+i}D_{s+i}$, where both C_k and D_k satisfy the same generalized second-order recurrence. They lead to many summation and product formulas for Fibonacci-type, Pell-type, and Jacobsthal-type numbers.

1. Introduction

Closed forms for the summations $\sum_{i=1}^{n} F_i^2$, $\sum_{i=0}^{n} L_i^2$, and $\sum_{i=0}^{n} F_iL_i$ are well-known, see, for example, [11, 13]. They prompted us to study the summation

$$\sum_{i=0}^{n} U_{r+i}V_{s+i},$$

where both sequences $\{U_k\}$ and $\{V_k\}$ satisfy the same recurrence with the Fibonacci numbers:

$$U_{k+2} = U_{k+1} + U_k,$$
$$V_{k+2} = V_{k+1} + V_k.$$

Normally, the recurrence would require $k \geq 0$. Nevertheless, such a restriction can be omitted, because we could push the recurrence backward so as to extend the subscripts to the negative integers. In effect, the entire sequences $\{U_k\}_{k=-\infty}^{\infty}$ and $\{V_k\}_{k=-\infty}^{\infty}$ satisfy the same recurrence. All we need is to define any two consecutive values. Naturally, we assume that the values of U_0, U_1, V_0, and V_1 are known. In fact, U_{-n} may be related to U_n in a rather simple manner. For instance, it is well-known, and can be easily proven by induction or via Binet's formulas, that $F_{-n} = (-1)^{n-1}F_n$, and $L_{-n} = (-1)^n L_n$.

Fibonacci-type recurrences have been studied extensively. They enjoy many fascinating properties; see, for example, [11, 13]. We found two simple formulas for the summation $\sum_{i=0}^{n} U_{r+i}V_{s+i}$. They led to many known and some new results. Encouraged by what we found, we attempted to extend them to Pell numbers [1, 12] and the accompanying Pell-Lucas numbers defined by

$$P_0 = 0, \quad P_1 = 1, \qquad P_{k+2} = 2P_{k+1} + P_k,$$
$$Q_0 = 2, \quad Q_1 = 2, \qquad Q_{k+2} = 2Q_{k+1} + Q_k.$$

Similar results were obtained. Next, we investigated the Jacobsthal numbers [7, 8, 9] and the associated Jacobsthal-Lucas numbers defined by

$$J_0 = 0, \quad J_1 = 1, \qquad J_{k+2} = J_{k+1} + 2J_k,$$
$$K_0 = 2, \quad K_1 = 1, \qquad K_{k+2} = K_{k+1} + 2K_k.$$

Interestingly, a simple shift of the coefficients made the problem harder. Nevertheless, we were able to obtain results that only required some slight modification. Ultimately, we found almost identical results for the generalized second-order recurrences, which will be discussed in Section 2. The special cases are studied in Section 3.

2. The Main Results

The generalized second-order recurrences have been studied extensively. See, for example, [4, 5, 10]. We focus our attention to the sum of products of any pair of generalized second-order recurrences:

$$C_{k+2} = pC_{k+1} + qC_k, \tag{2.1}$$
$$D_{k+2} = pD_{k+1} + qD_k, \tag{2.2}$$

where $p^2 + 4q \neq 0$. We find a surprisingly simple closed form.

Theorem 2.1. *For any integers r and s, and any nonnegative integer n,*

$$p\sum_{i=0}^{n} q^{n-i} C_{r+i} D_{s+i} = \begin{cases} C_{r+n} D_{r+n+1} - q^{n+1} C_r D_{s-1} & \text{if } n \text{ is even,} \\ C_{r+n+1} D_{s+n} - q^{n+1} C_r D_{s-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By alternately using the two recurrences (2.2) and (2.1), we can write the summation as

$$\begin{aligned} & p\sum_{i=0}^{n} q^{n-i} C_{r+i} D_{s+i} \\ &= q^n C_r \cdot pD_s + q^{n-1} \cdot pC_{r+1} \cdot D_{s+1} + \cdots + pC_{r+n} D_{s+n} \\ &= q^n C_r \big(D_{s+1} - qD_{s-1}\big) + q^{n-1}\big(C_{r+2} - qC_r\big) D_{s+1} \\ &\quad + q^{n-2} C_{r+2}\big(D_{s+3} - qD_{s+1}\big) + q^{n-3}\big(C_{r+4} - qC_{r+2}\big) D_{s+3} \\ &\quad + \cdots \\ &\quad + \begin{cases} C_{r+n}\big(D_{s+n+1} - qD_{s+n-1}\big) & \text{if } n \text{ is even} \\ qC_{r+n-1}\big(D_{s+n} - qD_{s+n-2}\big) + \big(C_{r+n+1} - qC_{r+n-1}\big) D_{s+n} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The desired result follows directly from this telescoping summation. □

Using a slightly different approach, we obtain another simple closed form.

Theorem 2.2. *For any integers r and s, and any nonnegative integer n,*

$$p\sum_{i=0}^{n} q^{n-i} C_{r+i} D_{s+i} = \begin{cases} C_{r+n+1} D_{s+n} - q^{n+1} C_{r-1} D_s & \text{if } n \text{ is even,} \\ C_{r+n} D_{s+n+1} - q^{n+1} C_{r-1} D_s & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is similar to that of Theorem 2.1, except that it alternates between the two recurrences (2.1) and (2.2). □

Example 2.3. For $p = q = 1$, we obtain the pair of numbers U_k and V_k. Let $r = s = 0$. When $U_k = V_k = F_k$, then, since $F_{-1} = 1$, Theorems 2.1 and 2.2 yield the well-known formula (see, for example, [11, 13])

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

In a similar manner, setting $U_k = V_k = L_k$, we obtain, along with $L_{-1} = -1$,

$$\sum_{i=0}^{n} L_i^2 = L_n L_{n+1} + 2.$$

If we set $U_k = F_k$ and $V_k = L_k$, while Theorem 2.1 gives

$$\sum_{i=0}^{n} F_i L_i = \begin{cases} F_n L_{n+1} & \text{if } n \text{ is even,} \\ F_{n+1} L_n & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.2 produces a slightly different formula

$$\sum_{i=0}^{n} F_i L_i = \begin{cases} F_{n+1} L_n - 2 & \text{if } n \text{ is even,} \\ F_n L_{n+1} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Comparing these two results, we conclude that

$$F_n L_{n+1} - F_{n+1} L_n = \begin{cases} -2 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

This suggests a more general result can be derived from the summation formulas stated in the two main theorems. □

Corollary 2.4. *For any integers r and s,*

$$C_{r+n} D_{s+n+1} - C_{r+n+1} D_{s+n} = (-q)^{n+1} \big(C_{r-1} D_s - C_r D_{s-1} \big).$$

Proof. We could obtain the result by comparing the two formulas stated in Theorems 2.1 and 2.2. Alternatively, we note that

$$\begin{aligned} C_i D_{j+1} - C_{i+1} D_j &= C_i \big(p D_j + q D_{j-1} \big) - \big(p C_i + q C_{i-1} \big) D_j \\ &= -q \big(C_{i-1} D_j - C_i D_{j-1} \big), \end{aligned}$$

a repeated application of which yields the result stated in the corollary. □

The d'Ocagne's identity

$$F_n F_{m+1} - F_{n+1} F_m = (-1)^{n+1} F_{m-n}$$

is famous for its connection to a geometric puzzle (see, for example, [3]) that is often credited to Lewis Carroll, whose real name was Charles Lutwidge Dodgson, the author of *Alice's Adventures in Wonderland*. Our next result, which is obtained by setting $r = 0$ and $s = m - n$, can be regarded as the d'Ocagne's identity for any two generalized second-order recurrences.

Corollary 2.5. *For any integers m and n,*

$$C_n D_{m+1} - C_{n+1} D_m = (-q)^{n+1} \big(C_{-1} D_{m-n} - C_0 D_{m-n-1} \big).$$

The counterparts of Fibonacci and Lucas numbers within the family of generalized second-order recurrences are

$$\begin{aligned} &X_0 = 0, \quad X_1 = 1, \qquad X_{k+2} = p X_{k+1} + q X_k, \\ &Y_0 = 2, \quad Y_1 = p, \qquad Y_{k+2} = p Y_{k+1} + q Y_k, \end{aligned}$$

where $p^2 + 4q \neq 0$. The Binet's formulas for them are precisely what we expect from any sequences similar to Fibonacci and Lucas numbers:

$$X_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \text{and} \qquad Y_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}$, and $\beta = \frac{p - \sqrt{p^2 + 4q}}{2}$. The next result can be found in, among others, [4, 10].

Theorem 2.6. *For any integer n,*

$$\begin{aligned} X_{-n} &= (-1)^{n-1}\,\frac{X_n}{q^n}, \\ Y_{-n} &= (-1)^n\,\frac{Y_n}{q^n}, \\ qX_{n-1} + X_{n+1} &= Y_n, \\ qY_{n-1} + Y_{n+1} &= (p^2+4q)X_n. \end{aligned}$$

Proof. Since $\alpha\beta = -q$, we see that

$$\alpha^{-n} \pm \beta^{-n} = \frac{(-1)^n(\beta^n \pm \alpha^n)}{q^n},$$

Hence, $X_{-n} = (-1)^{n-1}X_n/q^n$, and $Y_{-n} = (-1)^n Y_n/q^n$. We also find

$$\begin{aligned} q(\alpha^{n-1} \pm \beta^{n-1}) + (\alpha^{n+1} \pm \beta^{n+1}) &= -\alpha\beta(\alpha^{n-1} \pm \beta^{n-1}) + (\alpha^{n+1} \pm \beta^{n+1}) \\ &= (\alpha-\beta)(\alpha^n \mp \beta^n). \end{aligned}$$

Therefore, $qX_{n-1} + X_{n+1} = Y_n$, and $qY_{n-1} + Y_{n+1} = (\alpha-\beta)^2 X_n = (p^2+4q)X_n$. □

Using Corollary 2.5, we obtain a collection of interesting identities. See [6] for other related results.

Corollary 2.7. *The following identities hold for any integers m and n:*

$$X_nX_{m+1} - X_{n+1}X_m = -(-q)^n X_{m-n}, \tag{2.3}$$

$$Y_nY_{m+1} - Y_{n+1}Y_m = (p^2+4q)(-q)^n X_{m-n}, \tag{2.4}$$

$$X_nY_{m+1} - X_{n+1}Y_m = -(-q)^n Y_{m-n}, \tag{2.5}$$

$$Y_nC_m = C_{m+n} + (-q)^n C_{m-n}, \tag{2.6}$$

$$(p^2+4q)X_nC_m = \big(C_{m+n+1} + qC_{m+n-1}\big) - (-q)^n\big(C_{m-n+1} + qC_{m-n-1}\big), \tag{2.7}$$

$$Y_nX_m = X_{m+n} + (-q)^n X_{m-n}, \tag{2.8}$$

$$Y_nY_m = Y_{m+n} + (-q)^n Y_{m-n}, \tag{2.9}$$

$$(p^2+4q)X_nX_m = Y_{m+n} - (-q)^n Y_{m-n}. \tag{2.10}$$

Proof. By letting $C_k = D_k = X_k$ in Corollary 2.5, together with $X_{-1} = \frac{1}{q}$, and $X_0 = 0$, we obtain the d'Ocagne's identity (2.3). Similarly, by setting $C_k = D_k = Y_k$, and recall that $Y_{-1} = -\frac{p}{q}$, and $Y_0 = 2$, we find

$$\begin{aligned} Y_nY_{m+1} - Y_{n+1}Y_m &= (-q)^{n+1}(Y_{-1}Y_{m-n} - Y_0Y_{m-n-1} \\ &= (-q)^{n+1}\big(-\tfrac{p}{q}\,Y_{m-n} - 2Y_{m-n-1}\big) \\ &= (-q)^n\big(pY_{m-n} + 2qY_{m-n-1}\big) \\ &= (-q)^n\big[(Y_{m-n+1} - qY_{m-n-1}) + 2qY_{m-n-1}\big] \\ &= (-q)^n\big(Y_{m-n+1} + qY_{m-n-1}\big) \\ &= (p^2+4q)(-q)^n X_{m-n}. \end{aligned}$$

This proves (2.4). Letting $C_k = X_k$ and $D_k = Y_k$ in Corollary 2.5 yields (2.5).

The product formula for Y_k and C_k in (2.6) can be derived as follows. We start with the following special case of Corollary 2.5:

$$X_n C_{m+1} - X_{n+1} C_m = -(-q)^n C_{m-n}. \tag{2.11}$$

By replacing n with $-n$, this becomes

$$X_{-n} C_{m+1} - X_{-n+1} C_m = -\left(-\frac{1}{q}\right)^n C_{m+n}.$$

Since $X_{-n} = (-1)^{n-1} \frac{X_n}{q^n}$, this reduces to

$$X_n C_{m+1} + q X_{n-1} C_m = C_{m+n}. \tag{2.12}$$

We note that the special case of $F_n U_{m+1} + F_{n-1} U_m = U_{m+n}$ also appeared in [2]. Subtracting (2.11) from (2.12), and applying the identity $qX_{n-1} + X_{n+1} = Y_n$, yields the desired result.

The product formula for X_k and C_k in (2.7) is more complicated. From Corollary 2.5, we also find

$$\begin{aligned} Y_n C_{m+1} - Y_{n+1} C_m &= (-q)^{n+1} \big(-\tfrac{p}{q} C_{m-n} - 2C_{m-n-1} \big) \\ &= (-q)^n \big(pC_{m-n} + 2qC_{m-n-1} \big) \\ &= (-q)^n \big[(C_{m-n+1} - qC_{m-n-1}) + 2qC_{m-n-1} \big] \\ &= (-q)^n \big(C_{m-n+1} + qC_{m-n-1} \big). \end{aligned}$$

Replacing n with $-n$ yields

$$Y_{-n} C_{m+1} - Y_{-n+1} C_m = \left(-\frac{1}{q}\right)^n \big(C_{m+n+1} + qC_{m+n-1} \big).$$

Since $Y_{-n} = (-1)^n \frac{Y_n}{q^n}$, the last equation becomes

$$Y_n C_{m+1} + q Y_{n-1} C_m = C_{m+n+1} + qC_{m+n-1}.$$

Subtraction yields

$$\big(qY_{n-1} + Y_{n+1} \big) C_m = \big(C_{m+n+1} + qC_{m+n-1} \big) - (-q)^n \big(C_{m-n+1} + qC_{m-n-1} \big).$$

The result follows from the identity $Y_{n-1} + Y_{n+1} = (p^2 + 4q) X_n$.

By letting C_m be X_m and Y_m, respectively, in (2.6), we obtain the product formulas (2.8) and (2.9). The product formula for $X_n X_m$ looks slightly different in (2.10). It is derived from (2.7) by letting $C_m = X_m$. □

3. Special Cases

To be able to use Corollaries 2.5 and 2.7, it is important to remember that the recurrences must all share the same coefficients p and q. When $p = q = 1$, we have $X_k = F_k$, and $Y_k = L_k$. Corollary 2.7 becomes the following.

Corollary 3.1. *The following identities hold for any integers m and n:*

$$\begin{aligned}
F_nF_{m+1} - F_{n+1}F_m &= -(-1)^nF_{m-n},\\
L_nL_{m+1} - L_{n+1}L_m &= 5(-1)^nF_{m-n},\\
F_nL_{m+1} - F_{n+1}L_m &= -(-1)^nL_{m-n},\\
L_nU_m &= U_{m+n} + (-1)^nU_{m-n},\\
5F_nU_m &= \big(U_{m+n+1} + U_{m+n-1}\big) - (-1)^n\big(U_{m-n+1} + U_{m-n-1}\big),\\
L_nF_m &= F_{m+n} + (-1)^nF_{m-n},\\
L_nL_m &= L_{m+n} + (-1)^nL_{m-n},\\
5F_nF_m &= L_{m+n} - (-1)^nL_{m-n}.
\end{aligned}$$

For $p = 2$, and $q = 1$, together with the initial values $X_0 = 0$, $X_1 = 1$, $Y_0 = Y_1 = 2$, we obtain the Pell and Pell-Lucas numbers P_n and Q_n, respectively (see Section 1).

Corollary 3.2. *The following identities hold for any integers m and n, and for any sequence A_k that satisfies the recurrence relation $A_{k+2} = 2A_{k+1} + A_k$:*

$$\begin{aligned}
P_nP_{m+1} - P_{n+1}P_m &= -(-1)^nP_{m-n},\\
Q_nQ_{m+1} - Q_{n+1}Q_m &= 8(-1)^nP_{m-n},\\
P_nQ_{m+1} - P_{n+1}Q_m &= -(-1)^nQ_{m-n},\\
Q_nA_m &= A_{m+n} + (-1)^nA_{m-n},\\
8P_nA_m &= \big(A_{m+n+1} + A_{m+n-1}\big) - (-1)^n\big(A_{m-n+1} + A_{m-n-1}\big),\\
Q_nP_m &= P_{m+n} + (-1)^nP_{m-n},\\
Q_nQ_m &= Q_{m+n} + (-1)^nQ_{m-n},\\
8P_nP_m &= Q_{m+n} - (-1)^nQ_{m-n}.
\end{aligned}$$

For the Jacobsthal and Jacobsthal-Lucas numbers J_k and K_k, we need $p = 1$, and $q = 2$, along with the initial values $X_0 = 0$, $X_1 = 1$, $Y_0 = 2$, and $Y_1 = 1$ (see Section 1).

Corollary 3.3. *The following identities hold for any integers m and n, and for any sequence B_k that satisfies the recurrence relation $B_{k+2} = B_{k+1} + 2B_k$:*

$$\begin{aligned}
J_nJ_{m+1} - J_{n+1}J_m &= -(-2)^nJ_{m-n},\\
K_nK_{m+1} - K_{n+1}K_m &= 9(-2)^nJ_{m-n},\\
J_nK_{m+1} - J_{n+1}K_m &= -(-2)^nK_{m-n},\\
K_nB_m &= B_{m+n} + (-2)^nB_{m-n},\\
9J_nB_m &= \big(B_{m+n+1} + 2B_{m+n-1}\big) - (-2)^n\big(B_{m-n+1} + 2B_{m-n-1}\big),\\
K_nJ_m &= J_{m+n} + (-2)^nJ_{m-n},\\
K_nK_m &= K_{m+n} + (-2)^nK_{m-n},\\
9J_nJ_m &= K_{m+n} - (-2)^nK_{m-n}.
\end{aligned}$$

Acknowledgment

The authors are grateful to the anonymous referee, whose valuable suggestions led to the expansion of our investigation and improvement of the presentation of this exposition.

References

[1] M. Bicknell, *A primer on the Pell sequence and related sequences*, The Fibonacci Quarterly, **13.4** (1975), 345–349.

[2] P. Hilton, J. Pedersen and L. Vrancken, *On certain arithmetic properties of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **33.3** (1995), 211–217.

[3] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, The Fibonacci Association, Santa Clara, CA, 1979.

[4] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quarterly, **3.2** (1965), 161–176.

[5] A. F. Horadam, *Special properties of the sequence* $W_n(a, b; p, q)$, The Fibonacci Quarterly, **5.5** (1967), 424–434.

[6] A. F. Horadam, *Sum of products: an extension*, The Fibonacci Quarterly, **17.3** (1979), 248–250.

[7] A. F. Horadam, *Jacobsthal and Pell curves*, The Fibonacci Quarterly, **26.1** (1988), 79–83.

[8] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34.1** (1996), 40–54.

[9] E. Jacobsthal, *Fibonaccische Polynome und Kreisteilungsgleichungen*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, **17** (1919-1920), 43–57.

[10] D. Jarden, *Recurring Sequences*, Riveon Lematematika, Jerusalem, 1958.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.

[12] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math., **1** (1877), 184–240, 289–321.

[13] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, John Wiley and Sons, New York, 1989.

MSC2000: 11B39.

c/o A. A. Khan, Regional Office, Indian Overseas Bank, Ashoka Marg, Lucknow, India
E-mail address: manzar57ghazipur@yahoo.co.in

Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063
E-mail address: kwong@fredonia.edu