THE GOLDEN RATIO, FIBONACCI NUMBERS AND BBP-TYPE FORMULAS

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Abstract. We derive interesting arctangent identities involving the golden ratio, Fibonacci numbers and Lucas numbers. Binary BBP-type formulas for the arctangents of the odd powers of the golden ratio are also derived, for the first time in the literature. Finally we derive golden-ratio-base BBP-type formulas for some mathematical constants, including π, $\log 2$, $\log\phi$ and $\sqrt{2}\arctan 2$. The $\phi-$nary BBP-type formulas derived here are considerably simpler than similar results contained in earlier literature.

1. Introduction

This paper is concerned with the derivation of interesting arctangent identities connecting the golden ratio, Fibonacci numbers and the related Lucas numbers. Binary BBP-type formulas for arctangents of the odd powers of the golden ratio will be derived, as well as golden-ratio-base BBP-type formulas for some mathematical constants. We will also present a couple of base 5 BBP-type formulas for linear combinations of the arctangents of even powers of the golden ratio. The golden ratio, having the numerical value of $(\sqrt{5}+1)/2$ is denoted throughout this paper by ϕ. The Fibonacci numbers are defined, as usual, through the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$. The Lucas numbers are defined by $L_n = F_{n-1} + F_{n+1}$.

We shall often make use of the following algebraic properties of ϕ

$$\phi^2 = 1 + \phi, \tag{1.1a}$$

$$\sqrt{5} = 2\phi - 1, \tag{1.1b}$$

$$\phi - 1 = 1/\phi, \tag{1.1c}$$

$$\phi^n = \phi F_n + F_{n-1}, \tag{1.1d}$$

$$\phi^{-n} = (-1)^n(-\phi F_n + F_{n+1}), \tag{1.1e}$$

and

$$\phi^n = \phi^{n-1} + \phi^{n-2}. \tag{1.1f}$$

We will also need the following trigonometric identities

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \quad xy < 1 \tag{1.2a}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \quad xy > -1\,. \tag{1.2b}$$

Results for arctangent identities involving the Fibonacci numbers and related sequences may also be found in earlier references [7, 10, 9] and references therein, while results for $\phi-$nary BBP-type formulas may be found in references [4, 5, 6, 11].

2. Arctangents of the Odd Powers of the Golden Ratio

In this section we will present results for the arctangents of the odd powers of the golden ratio in terms of the arctangents of reciprocal Fibonacci and reciprocal Lucas numbers, as well as in terms of the arctangents of consecutive Fibonacci numbers.

2.1. Arctangent formulas involving reciprocal Fibonacci and reciprocal Lucas numbers.

Theorem 2.1. *For positive integers k,*

$$\tan^{-1}\phi^{2k-1} = 2\tan^{-1}1 - \frac{1}{2}\tan^{-1}\left(\frac{2}{L_{2k-1}}\right). \tag{2.1}$$

Proof. Choosing $x = 1/\phi^{2k-1} = y$ in equation (1.2a), we find

$$\begin{aligned} &2\tan^{-1}\frac{1}{\phi^{2k-1}} \\ &= \tan^{-1}\left(\frac{2}{\phi^{2k-1}-\phi^{-(2k-1)}}\right) \\ &= \tan^{-1}\left(\frac{2}{F_{2k-2}+F_{2k}}\right) \\ &= \tan^{-1}\left(\frac{2}{L_{2k-1}}\right), \end{aligned}$$

and the result follows. ☐

Theorem 2.2. *For non-zero integers k,*

$$\tan^{-1}\phi^{2k+1} = 2\tan^{-1}1 + \frac{1}{2}\tan^{-1}\left(\frac{1}{L_{2k}}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{F_{2k}}\right) \tag{2.2a}$$

$$\tan^{-1}\phi^{2k-1} = 2\tan^{-1}1 - \frac{1}{2}\tan^{-1}\left(\frac{1}{L_{2k}}\right) - \frac{1}{2}\tan^{-1}\left(\frac{1}{F_{2k}}\right) \tag{2.2b}$$

Proof. Choosing $x = 1/\phi^{2k-1}$ and $y = 1/\phi^{2k+1}$ in equation (1.2a) and using the algebraic properties of ϕ, it is straightforward to establish that

$$\begin{aligned} &\tan^{-1}\left(\frac{1}{\phi^{2k-1}}\right) + \tan^{-1}\left(\frac{1}{\phi^{2k+1}}\right) \\ &= \tan^{-1}\left(\frac{\phi+\phi^{-1}}{\phi^{2k}-\phi^{-2k}}\right) \\ &= \tan^{-1}\frac{\sqrt{5}}{\sqrt{5}F_{2k}} = \tan^{-1}\left(\frac{1}{F_{2k}}\right). \end{aligned}$$

That is

$$\tan^{-1}\phi^{2k+1} + \tan^{-1}\phi^{2k-1} = \pi - \tan^{-1}\left(\frac{1}{F_{2k}}\right). \tag{2.3}$$

Choosing $x = \phi^{2k+1}$ and $y = \phi^{2k-1}$ in equation (1.2b), we find

$$\begin{aligned}\tan^{-1}\phi^{2k+1} - \tan^{-1}\phi^{2k-1} &= \tan^{-1}\left(\frac{\phi-\phi^{-1}}{\phi^{-2k}+\phi^{2k}}\right)\\ &= \tan^{-1}\left(\frac{1}{L_{2k}}\right).\end{aligned} \tag{2.4}$$

Addition of equations (2.3) and (2.4) gives equation (2.2a), while subtraction of equation (2.4) from equation (2.3) gives equation (2.2b). □

Remark 2.1. *Note that by performing the telescoping summation suggested by equation* (2.4), *it is established that*

$$\tan^{-1}\phi^{2n+1} = \tan^{-1}\phi + \sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{L_{2k}}\right),$$

which can be written as

$$\tan^{-1}\phi^{2n+1} = \tan^{-1}1 + \frac{1}{2}\tan^{-1}\frac{1}{2} + \sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{L_{2k}}\right), \tag{2.5}$$

since

$$\tan^{-1}\phi = \tan^{-1}1 + \frac{1}{2}\tan^{-1}\frac{1}{2} \quad (k=1 \text{ in Theorem 2.1}). \tag{2.6}$$

2.2. Arctangent formulas involving the ratio of consecutive Fibonacci numbers.

Theorem 2.3. *For non-negative integers n,*

$$\tan^{-1}(\phi^{4n-1}) = 3\tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2} - \tan^{-1}\left(\frac{F_{2n-1}}{F_{2n}}\right), \tag{2.7a}$$

$$\tan^{-1}(\phi^{4n-3}) = \tan^{-1}1 + \frac{1}{2}\tan^{-1}\frac{1}{2} + \tan^{-1}\left(\frac{F_{2n-2}}{F_{2n-1}}\right). \tag{2.7b}$$

Proof. Choosing $x = 1/\phi$ and $y = F_{p-1}/F_p$ in the trigonometric identity (1.2b), clearing fractions and using properties (1.1d) and (1.1e) to simplify the numerator and denominator of the arctangent argument, and replacing $\tan^{-1}\phi$ with the right-hand side of equation (2.6), we obtain

$$(-1)^p\tan^{-1}(\phi^{2p-1}) = (2(-1)^p+1)\tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{F_{p-1}}{F_p}. \tag{2.8}$$

Setting $p = 2n$ in (2.8), we obtain identity (2.7a), while $p = 2n-1$ in (2.8) gives identity (2.7b). □

3. Arctangents of the Reciprocal Even Powers of the Golden Ratio

Theorem 3.1. *For non-negative integers k,*

$$2\tan^{-1}\left(\frac{1}{\phi^{2k}}\right) = \tan^{-1}\left(\frac{2}{F_{2k}\sqrt{5}}\right). \tag{3.1}$$

Proof. Equation (3.1) follows from the choice of $x = 1/\phi^{2k} = y$ in equation (1.2a). □

Theorem 3.2. *For non-negative integers k,*

$$2\tan^{-1}\left(\frac{1}{\phi^{2k}}\right)=\tan^{-1}\left(\frac{\sqrt{5}}{L_{2k+1}}\right)+\tan^{-1}\left(\frac{1}{F_{2k+1}\sqrt{5}}\right) \tag{3.2a}$$

$$2\tan^{-1}\left(\frac{1}{\phi^{2k+2}}\right)=\tan^{-1}\left(\frac{\sqrt{5}}{L_{2k+1}}\right)-\tan^{-1}\left(\frac{1}{F_{2k+1}\sqrt{5}}\right). \tag{3.2b}$$

Proof. Choosing $x=1/\phi^{2k}$ and $y=1/\phi^{2k+2}$ in equation (1.2a) and using the algebraic properties of ϕ, it is straightforward to establish that

$$\tan^{-1}\left(\frac{1}{\phi^{2k}}\right)+\tan^{-1}\left(\frac{1}{\phi^{2k+2}}\right)=\tan^{-1}\left(\frac{\sqrt{5}}{L_{2k+1}}\right). \tag{3.3}$$

Choosing $x=1/\phi^{2k}$ and $y=1/\phi^{2k+2}$ in equation (1.2b), we find

$$\tan^{-1}\left(\frac{1}{\phi^{2k}}\right)-\tan^{-1}\left(\frac{1}{\phi^{2k+2}}\right)=\tan^{-1}\left(\frac{1}{F_{2k+1}\sqrt{5}}\right). \tag{3.4}$$

Addition of equations (3.3) and (3.4) gives equation (3.2a), while subtraction of equation (3.4) from equation (3.3) gives equations (3.2b). □

Remark 3.1. *Performing the telescoping summation invited by equation* (3.4)*, we obtain*

$$\tan^{-1}\left(\frac{1}{\phi^{2}}\right)-\tan^{-1}\left(\frac{1}{\phi^{2n+2}}\right)=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{F_{2k+1}\sqrt{5}}\right). \tag{3.5}$$

Taking limit $n\to\infty$, we obtain the formula

$$\tan^{-1}\left(\frac{1}{\phi^{2}}\right)=\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{F_{2k+1}\sqrt{5}}\right). \tag{3.6}$$

4. Arctangent Identities Involving the Fibonacci and Lucas Numbers

4.1. **Arctangent formulas involving reciprocal Fibonacci and reciprocal Lucas numbers.**

Theorem 4.1. *For non-zero integers n,*

$$\tan^{-1}\left(\frac{1}{F_{2n-2}}\right)=\tan^{-1}\left(\frac{1}{F_{2n}}\right)+\tan^{-1}\left(\frac{1}{L_{2n-2}}\right)+\tan^{-1}\left(\frac{1}{L_{2n}}\right), \tag{4.1a}$$

$$\tan^{-1}\left(\frac{2}{L_{2n-1}}\right)=\tan^{-1}\left(\frac{1}{F_{2n}}\right)+\tan^{-1}\left(\frac{1}{L_{2n}}\right), \tag{4.1b}$$

$$\tan^{-1}\left(\frac{2}{L_{2n+1}}\right)=\tan^{-1}\left(\frac{1}{F_{2n}}\right)-\tan^{-1}\left(\frac{1}{L_{2n}}\right), \tag{4.1c}$$

$$\tan^{-1}\left(\frac{2}{F_{2n}\sqrt{5}}\right)=\tan^{-1}\left(\frac{\sqrt{5}}{L_{2n+1}}\right)+\tan^{-1}\left(\frac{1}{F_{2n+1}\sqrt{5}}\right). \tag{4.1d}$$

Proof. Equation (4.1a) is proved by setting $k = n - 1$ in equation (2.2a) and $k = n$ in equation (2.2b) and eliminating $\tan^{-1}(\phi^{2n-1})$ between the two equations that result. Equation (4.1b) is proved by setting $k = n$ in equation (2.1) and $k = n$ in equation (2.2b) and eliminating $\tan^{-1}(\phi^{2n-1})$ between the two equations that result. Equation (4.1c) is proved by setting $k = n + 1$ in equation (2.1) and $k = n$ in equation (2.2a) and eliminating $\tan^{-1}(\phi^{2n+1})$ between the two equations that result. Lastly, equation (4.1d) is proved by setting $k = n + 1$ in equation (3.2b) and $k = n$ in equation (3.2a) and eliminating $\tan^{-1}(1/\phi^{2n})$ between the resulting equations. □

Remark 4.1. *Using the identity* (*Theorem 4 of* [7]),

$$\tan^{-1}\left(\frac{1}{F_{2n}}\right) = \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+2}}\right), \tag{4.2}$$

equations (4.1a)–(4.1c) *can also be written*

$$\tan^{-1}\left(\frac{1}{F_{2n-1}}\right) = \tan^{-1}\left(\frac{1}{L_{2n-2}}\right) + \tan^{-1}\left(\frac{1}{L_{2n}}\right), \tag{4.3a}$$

$$\tan^{-1}\left(\frac{2}{L_{2n-1}}\right) = \tan^{-1}\left(\frac{1}{L_{2n}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+2}}\right), \tag{4.3b}$$

$$\tan^{-1}\left(\frac{2}{L_{2n+1}}\right) = \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+2}}\right) - \tan^{-1}\left(\frac{1}{L_{2n}}\right). \tag{4.3c}$$

Equation (4.3a) *is Theorem 3 of* [7].

Subtracting equation (4.1c) from equation (4.1b) we obtain, for non-zero integers, the following arctangent identity involving three consecutive Lucas numbers.

Theorem 4.2.

$$\tan^{-1}\left(\frac{2}{L_{2n-1}}\right) = 2\tan^{-1}\left(\frac{1}{L_{2n}}\right) + \tan^{-1}\left(\frac{2}{L_{2n+1}}\right). \tag{4.4}$$

Remark 4.2. *It is instructive to compare the two identities, equation* (4.2) *involving Fibonacci numbers and equation* (4.4) *involving Lucas numbers.*

4.2. **Arctangent formulas involving the ratio of consecutive Fibonacci numbers.**

Theorem 4.3. *For positive integers n,*

$$\tan^{-1}\left(\frac{F_{2n}}{F_{2n+1}}\right) = \sum_{k=1}^{2n}\tan^{-1}\left(\frac{1}{L_{2k}}\right), \tag{4.5a}$$

$$\tan^{-1}\left(\frac{F_{2n-1}}{F_{2n}}\right) = \tan^{-1}2 - \sum_{k=1}^{2n-1}\tan^{-1}\left(\frac{1}{L_{2k}}\right), \tag{4.5b}$$

$$\tan^{-1}\left(\frac{F_{2n}}{F_{2n+1}}\right) = \tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2}{L_{4n+1}}\right) \tag{4.5c}$$

and

$$\tan^{-1}\left(\frac{F_{2n-1}}{F_{2n}}\right) = \tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2} + \frac{1}{2}\tan^{-1}\left(\frac{2}{L_{4n-1}}\right). \tag{4.5d}$$

Proof. Comparing identities (2.7b) and (2.5), we obtain identity (4.5a), expressing, as a sum of reciprocal arctangents of even indexed Lucas numbers, the arctangent of the ratio of any two consecutive Fibonacci numbers, with the even indexed Fibonacci number as the numerator.

Similarly, by comparing identities (2.7a) and (2.5), we obtain identity (4.5b), expressing, as a sum of the arctangents of reciprocal even indexed Lucas numbers, the arctangent of the ratio of any two consecutive Fibonacci numbers, with the odd indexed Fibonacci number as the numerator. Equation (4.5c) follows from equation (2.1) and equation (2.7b) while equation (4.5d) is obtained by comparing equation (2.1) and equation (2.7a). □

Taking limit $n \to \infty$ in equation (4.5a), we obtain the following theorem.

Theorem 4.4.

$$\tan^{-1}\frac{1}{\phi} = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{L_{2k}}\right). \tag{4.6}$$

Remark 4.3. *It is instructive to compare equation* (4.5a) *with the well-known result*

$$\tan^{-1}\frac{1}{F_{2n}} = \sum_{k=n}^{\infty} \tan^{-1}\left(\frac{1}{F_{2k+1}}\right), \tag{4.7}$$

and to compare (4.6) *with the case $n = 1$ in equation* (4.7)*, namely,*

$$\tan^{-1} 1 = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2k+1}}\right). \tag{4.8}$$

Equation (4.6) *was also proved in* [7] *(Theorem 6).*

5. BBP-type Formulas

The convergent series

$$C = \sum_{k\geq 0} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s} \equiv P(s,b,l,A), \tag{5.1}$$

where s and l are integers, b is a real number, and $A = (a_1, a_2, \ldots, a_l)$ is a vector of real numbers, defines a base-b expansion of the polylogarithm constant C. If b is an integer and A is a vector of integers, then equation (5.1) is called a BBP-type formula for the mathematical constant C. A BBP-type formula has the remarkable property that it allows the ith digit of a mathematical constant to be computed without having to compute any of the previous $i-1$ digits and without requiring ultra high-precision [8, 2]. BBP-type formulas were first introduced in a 1996 paper [3], where a formula of this type for π was given. The BBP-type formulas derived in this section will be given in the standard notation, defined by equation (5.1).

5.1. **Binary BBP-type formulas for the arctangents of odd powers of the golden ratio.** Identities (2.1), (2.2a), (2.2b), (2.7a) and (2.7b) give binary BBP-type formulas for the odd powers of ϕ, whenever binary BBP-type formulas exist for the rational numbers whose arctangents are involved. The first few BBP-type series ready examples are the following:

$$\tan^{-1}\phi = \tan^{-1} 1 + \frac{1}{2}\tan^{-1}\frac{1}{2} \quad \text{(equation (2.6))}, \tag{5.2}$$

$$\tan^{-1}\phi^3 = 2\tan^{-1} 1 - \frac{1}{2}\tan^{-1}\frac{1}{2} \quad \text{(n=1 in equation (2.7a))}, \tag{5.3}$$

$$\tan^{-1}\phi^5 = \tan^{-1}1 + \frac{3}{2}\tan^{-1}\frac{1}{2} \quad \text{(n=2 in equation (2.7b))}, \tag{5.4}$$

$$\tan^{-1}\phi^7 = 3\tan^{-1}1 - \frac{3}{2}\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{8} \quad \text{(n=2 in equation (2.7a))}, \tag{5.5}$$

and

$$\tan^{-1}\phi^9 = 2\tan^{-1}1 + \frac{1}{2}\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4} \quad \text{(n=3 in equation (2.7b))}. \tag{5.6}$$

In obtaining the final form of equation (5.6) we used

$$\tan^{-1}\frac{3}{5} = \tan^{-1}1 - \tan^{-1}\frac{1}{4}.$$

To derive the BBP-type formulas that correspond to equations (5.2)—(5.6), we will employ the following BBP-type formulas in general bases, derived in reference [1]:

$$\tan^{-1}\frac{1}{u} = \frac{1}{u^3}P(1, u^4, 4, (u^2, 0, -1, 0)), \tag{5.7}$$

$$\tan^{-1}\left(\frac{1}{2u-1}\right) = \frac{1}{16u^7}P(1, 16u^8, 8, (8u^6, 8u^5, 4u^4, 0, -2u^2, -2u, -1, 0)) \tag{5.8}$$

and

$$\tan^{-1}\left(\frac{1}{2u+1}\right) = \frac{1}{16u^7}P(1, 16u^8, 8, (8u^6, -8u^5, 4u^4, 0, -2u^2, 2u, -1, 0)). \tag{5.9}$$

Using $u = 2$ in equation (5.7) and $u = 1$ in equation (5.8), and forming the indicated linear combinations, equations (5.2)—(5.4) give rise to the following BBP-type formulas:

$$\tan^{-1}\phi = \frac{1}{16}P(1, 16, 8, (8, 16, 4, 0, -2, -4, -1, 0)), \tag{5.10}$$

$$\tan^{-1}(\phi^3) = \frac{1}{8}P(1, 16, 8, (8, 4, 4, 0, -2, -1, -1, 0)), \tag{5.11}$$

and

$$\tan^{-1}(\phi^5) = \frac{1}{16}P(1, 16, 8, (8, 32, 4, 0, -2, -8, -1, 0)). \tag{5.12}$$

Using $u = 1$ in equation (5.8) and $u = 2$ in equation (5.7) and expanding both series to base 2^{12}, length 24, and using $u = 8$ in equation (5.7) and finally forming the indicated linear combination gives the BBP-type formula for $\tan^{-1}\phi^7$ as

$$\begin{aligned}\tan^{-1}(\phi^7) = \frac{3}{4096}P(1, 2^{12}, 24, (&2048, 0, 1024, 0, -512, -1024, -256,\\ &0, 128, 0, 64, 0, -32, 0, -16, 0, 8, 16, 4, 0, -2, 0, -1, 0)).\end{aligned} \tag{5.13}$$

Using $u = 1$ in equation (5.8), $u = 2$ in equation (5.7) and $u = 4$ in equation (5.7), expanding the three series to base 256, length 16, and forming the indicated linear combination in equation (5.6) gives the BBP-type formula for $\tan^{-1}\phi^9$ as

$$\tan^{-1}(\phi^9) = \frac{1}{128}P(1,256,16,(128,192,64,-128,-32,-48, \\ -16,0,8,12,4,8,-2,-3,-1,0)). \tag{5.14}$$

5.2. **Base 5 BBP-type formulas.** From the identities (3.1), (3.2a) and (3.2b) we can form the following BBP-type series ready combinations

$$\tan^{-1}\left(\frac{1}{\phi^2}\right) + \tan^{-1}\left(\frac{1}{\phi^4}\right) = \tan^{-1}\left(\frac{\sqrt{5}}{4}\right) \\ = \tan^{-1}\left(\frac{1}{\sqrt{5}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{5}^3}\right), \tag{5.15}$$

$$\tan^{-1}\left(\frac{1}{\phi^2}\right) + \tan^{-1}\left(\frac{1}{\phi^6}\right) = \tan^{-1}\left(\frac{1}{\sqrt{5}}\right), \tag{5.16}$$

and

$$\tan^{-1}\left(\frac{1}{\phi^4}\right) - \tan^{-1}\left(\frac{1}{\phi^6}\right) = \tan^{-1}\left(\frac{1}{\sqrt{5}^3}\right). \tag{5.17}$$

According to equation (5.7),

$$\sqrt{5}\tan^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{5}P(1,25,4,(5,0,-1,0)), \tag{5.18}$$

and

$$\sqrt{5}\tan^{-1}\left(\frac{1}{\sqrt{5}^3}\right) = \frac{1}{5^4}P(1,5^6,4,(5^3,0,-1,0)). \tag{5.19}$$

Therefore, equations (5.15)—(5.17) give rise to the following base 5 BBP-type formulas:

$$\sqrt{5}\left\{\tan^{-1}\left(\frac{1}{\phi^2}\right) + \tan^{-1}\left(\frac{1}{\phi^4}\right)\right\} \\ = \frac{1}{5^5}P(1,5^6,12,(5^5,0,2\cdot 5^4,0,5^3,0,-5^2,0,-10,0,-1,0)), \tag{5.20}$$

$$\sqrt{5}\left\{\tan^{-1}\left(\frac{1}{\phi^2}\right) + \tan^{-1}\left(\frac{1}{\phi^6}\right)\right\} = \frac{1}{5}P(1,25,4,(5,0,-1,0)), \tag{5.21}$$

and

$$\sqrt{5}\left\{\tan^{-1}\left(\frac{1}{\phi^4}\right) - \tan^{-1}\left(\frac{1}{\phi^6}\right)\right\} = \frac{1}{5^4}P(1,5^6,4,(5^3,0,-1,0)). \tag{5.22}$$

5.3. **BBP-type formulas in base ϕ.** Many BBP-type formulas in general bases were derived in reference [1]. Base ϕ formulas are easily obtained by choosing the base in any general formula of interest to be a power of ϕ, and using the algebraic properties of ϕ. We note that since ϕ is not an integer, these series are, technically speaking, not BBP-type, in the sense that they do not really lead to any digit extraction formulas, but rather correspond to base ϕ expansions of the mathematical constants concerned. We now present some interesting degree 1 base ϕ formulas. ϕ-nary BBP-type formulas for π were also derived in references [4, 6, 11]. The formulas presented here are considerably simpler and more elegant than those found in the earlier papers.

By setting $n = \phi$ in equation 27 of [1] we obtain a $\phi-$nary BBP-type formula for π:

$$\pi = \frac{4}{\phi^5} P(1, -\phi^6, 6, (\phi^4, 0, 2\phi^2, 0, 1, 0)). \tag{5.23}$$

The base ϕ^{12}, length 12 version of equation (5.23) is

$$\pi = \frac{4}{\phi^{11}} P(1, \phi^{12}, 12, (\phi^{10}, 0, 2\phi^8, 0, \phi^6, 0, -\phi^4, 0, -2\phi^2, 0, -1, 0)). \tag{5.24}$$

The following $\phi-$nary formulas are also readily obtained:

$$\log\phi = \frac{1}{\phi^2} P(1, \phi^2, 2, (\phi, -1)) \quad (n = \phi \text{ in equation 28 of [1]}), \tag{5.25}$$

$$\log 2 = \frac{1}{\phi^3} P(1, \phi^3, 3, (\phi^2, \phi, -2)) \quad (n = \phi \text{ in equation 33 of [1]}), \tag{5.26}$$

$$\arctan(\frac{1}{\phi}) = \frac{1}{\phi^3} P(1, \phi^4, 4, (\phi^2, 0, -1, 0)) \quad (u = \phi \text{ in equation 8 of [1]}), \tag{5.27}$$

$$\sqrt{3}\arctan\left(\sqrt{\frac{3}{5}}\right) = \frac{3}{2\phi^5} P(1, \phi^6, 6, (\phi^4, \phi^3, 0, -\phi, -1, 0)) \quad (n = \phi \text{ in eq. 12 of [1]}), \tag{5.28}$$

$$\sqrt{3}\arctan\left(\frac{\sqrt{3}}{\phi^3}\right) = \frac{3}{2\phi^2} P(1, \phi^3, 3, (\phi, -1, 0)) \quad (n = \phi \text{ in equation 13 of [1]}), \tag{5.29}$$

$$\arctan\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{16\phi^7} P(1, 16\phi^8, 8, (8\phi^6, 8\phi^5, 4\phi^4, 0, -2\phi^2, -2\phi, -1, 0)) \quad (n = \phi \text{ in equation 17 of [1]}), \tag{5.30}$$

$$\arctan\left(\frac{1}{\phi^3}\right) = \frac{1}{16\phi^7} P(1, 16\phi^8, 8, (8\phi^6, -8\phi^5, 4\phi^4, 0, -2\phi^2, 2\phi, -1, 0)) \quad (n = \phi \text{ in equation 18 of [1]}), \tag{5.31}$$

$$\sqrt{2}\arctan\sqrt{2} = \frac{2}{\phi^7} P(1, \phi^8, 8, (\phi^6, 0, \phi^4, 0, -\phi^2, 0, -1, 0)) \quad (\text{n=}\phi^2 \text{ in equation 21 of [1]}), \tag{5.32}$$

$$27\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) = \frac{3}{2\phi^5} P(1, -27\phi^6, 6, (9\phi^4, 9\phi^3, 6\phi^2, 3\phi, 1, 0)) \quad (n = \phi \text{ in eq. 25 of [1]}), \tag{5.33}$$

and

$$27\sqrt{3}\,\arctan\left(\frac{1}{\phi^3\sqrt{3}}\right) = \frac{3}{2\phi^5}P(1,-27\,\phi^6,6,(9\,\phi^4,-9\,\phi^3,6\,\phi^2,-3\,\phi,1,0)) \tag{5.34}$$

$(n = \phi$ in eq. 26 of [1]).

6. Conclusion

We have derived and presented interesting arctangent identities connecting the golden ratio, Fibonacci numbers and the Lucas numbers. Binary BBP-type formulas for the arctangents of the odd powers of the golden ratio and base 5 formulas for combinations of the arctangents of the reciprocal even powers of the golden ratio were derived. We also presented results for the $\phi-$nary expansion of some mathematical constants.

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