PRODUCTS AND POWERS, POWERS AND EXPONENTIATIONS, ...

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ABSTRACT. The Horadam recurrence relation $w_{n+1}(a,b;p,q) = pw_n(a,b;p,q) - qw_{n-1}(a,b;p,q)$ (with $w_0 = a$ and $w_1 = b$) has inspired consideration of the recurrence $z_n(a,b;p,q) = z_n^p(a,b;p,q).z_{n-1}^q$ (with $z_0 = a$ and $z_1 = b$). This paper defines a natural sequence of such recurrence relations of which w_n and z_n are the first and second.

1. THE FUNCTIONS $w_n(a,b;p,q)$ AND $z_n(a,b;p,q)$

The Horadam functions [6, p. 161] and the functions $z_n(a,b;p,q)$ (Bunder [2, p. 279] and Larcombe and Bagdasar [8]) are given by:

Definition 1.1. *Let* $w_0(a,b;p,q) = a$, $w_1(a,b;p,q) = b$, *and for* $n \geq 1$ *let* $w_{n+1}(a,b;p,q) = pw_n(a,b;p,q) - qw_{n-1}(a,b;p,q)$.

Definition 1.2. *Let* $z_0(a,b;p,q) = a$, $z_1(a,b;p,q) = b$, *and for* $n \geq 1$ *let* $z_{n+1}(a,b;p,q) = (z_n(a,b;p,q))^p \cdot (z_{n-1}(a,b;p,q))^q$.

The Horadam functions $w_n(a,b;p,q)$ will usually be written as w_n and $z_n(a,b;p,q)$ will be written as z_n.

2. A SEQUENCE OF FUNCTIONS STARTING WITH w_n AND z_n

The Horadam recurrence of Definition 1.1 involves the sum of two products (i.e. repeated additions) pw_n and $(-q)w_{n-1}$. The recurrence in Definition 1.2 involves the product of two powers (i.e. repeated multiplications) z_n^p and z_{n-1}^q. Taking this to the next level, the recurrence would involve the exponentiation of repeated exponentiations

$$\left(t_n^{\cdot^{\cdot^{\cdot^{t_n}}}}\right) \quad \text{and} \quad \left(t_{n-1}^{\cdot^{\cdot^{\cdot^{t_{n-1}}}}}\right),$$

where there are p t_n's and q t_{n-1}'s. There are of course two different exponentiations, but we will consider only one.

The first aim of this paper is to generate a natural infinite sequence of such functions $< w_n, z_n, t_n, \dots >$ and the second to see whether t_n and later functions can be defined in simple terms or in terms of functions coming earlier in the sequence, just as z_n can be defined in terms of w_n. Bunder [2] and Larcombe and Bagdasar [8] show that

$$z_n = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)}.$$

The first aim can be achieved by using the following function due to Ackermann [1].

Definition 2.1. *Let* m *and* n *be positive integers. Define*

$$\phi(m,n,0) = m+n, \quad \phi(m,0,1) = 0, \quad \phi_(m,0,2) = 1, \quad \phi(m,0,r) = m, \quad \textit{for} \ \ r > 2,$$
$$\phi(m,n,r) = \phi(m,\phi(m,n-1,r),r-1), \quad \textit{for} \ \ n > 0, r > 0.$$

This gives $\phi(m,n,1) = mn$, $\phi(m,n,2) = m^n$, $\phi(m,n,3) = m^{\cdot^{\cdot^{m}}}$ (n m's).

Ackermann considered such functions to clarify Hilbert's proposed proof of the continuum hypothesis. It is also one of the earliest and simplest examples of a total function that is computable but not primitive recursive (see van Heijenoort [9]). The function $\phi(m,n,3)$, often written as ${}^n m$ was already known to Euler. The Ackermann function $\phi(m,n,r)$ is sometimes written as $ack(m,n,r)$, for example, see Giesler [5]. Knuth [7] and Conway and Guy [4] have other notations for the ϕ or ack function.

Note that Ackermann's $\phi(m,n,r)$ is related to, but not the same as, what is these days usually called the Ackermann function.

3. A General Horadam-Style Recurrence

A general Horadam recurrence, motivated by the discussion in Section 1, is given by the following definition.

Definition 3.1. *Let a, b, p, and q be integers. Let $s_{i,0}(a,b;p,q) = a$, $s_{i,1}(a,b;p,q) = b$, and for $n \geq 1$ let $s_{i,n+1}(a,b;p,q) = \phi(\phi(s_{i,n}(a,b;p,q),p,i+1),\phi(s_{i,n-1}(a,b;p,q),q,i+1),i)$.*

We will usually write $s_{i,n}(a,b;p,q)$ as $s_{i,n}$.

Clearly,

$$s_{1,n} = w_n(a,b;p,-q), s_{2,n} = z_n \text{ and } s_{3,n+1} = \left(s_{3,n}^{\cdot^{\cdot^{s_{3,n}}}}\right)^{\left(s_{3,n-1}^{\cdot^{\cdot^{s_{3,n-1}}}}\right)},$$

where there are p '$s_{3,n}$'s and q $s_{3,n-1}$'s.

Unless the meaning of repeated exponentiation can somehow be generalized, this, of course, requires p and q to be positive integers.

(Note that our notation would have been neater, given $w_n = s_{1,n}$, if we had q for $-q$ on the right-hand side of the recurrence in Definition 1.1, as this gives $s_{1,n} = w_n$!)

4. $s_{m,n}$ in Simple Terms or in Terms of $s_{j,n}$ where $j < m$

We first note that$s_{1,n}$ can be expressed in the following way. If

$$n \geq 0,\ p^2 \neq -4q,\ C = (p + \sqrt{(p^2+4q)})/2 \text{ and } D = (p + \sqrt{(p^2+4q)})/2,$$

then

$$s_{1,n} = \left(\frac{b - aC}{C - D}\right) C^n + \left(\frac{b - aD}{D - C}\right) D^n.$$

If $n \geq 0$, then

$$s_{1,n}(a,b,p,-p^2/4) = nb(p/2)^{n-1} - (n-1)a(p/2)^n.$$

For a reference see [6, pp. 161, 175] and Bunder [3].

In Section 2, $s_{2,n}(= z_n)$ was given in terms of $w_n(0,1;p,-q)$ and $w_n(1,0;p,-q)$, we also have:

$$s_{1,n} = w_n(a,b;p,-q) = aw_n(1,0;p,-q) + bw_n(0,1;p,-q),$$

so we might expect

$$s_{3,n} = \left(b^{\cdot^{\cdot^{b}}}\right)^{\left(a^{\cdot^{\cdot^{a}}}\right)},$$

where there are $w_n(0,1;p,-q)$ b's and $w_n(1,0;p,-q)$ a's. However the examples below show that this is not generally the case. Even in simple cases such as $i = 3$ and $p, q < 5$, there seems to be no simple expressions for $s_{i,n}$, nor one in terms of $s_{j,n}$ where $j < i$.

Example 4.1. *If* $p = -q = 1$ *then*

$$< w_n > = < s_{1,n} > = < a, b, a+b, a+2b, 2a+3b, \ldots >$$

$$< z_n > = < s_{2,n} > = < a, b, ab, ab^2, a^2b^3, \ldots >$$

$$s_{3,n+1} = s_{3,n}^{s_{3,n-1}} \quad \textit{and} \quad < s_{3,n} > = < a, b, b^a, b^{ab}, b^{ab^{1+a}}, b^{ab^{1+a+ab+ab^{1+a}}}, \ldots > .$$

Example 4.2. *If* $p = 3$, $q = -2$, *then*

$$< w_n > = < s_{1,n} > = < a, b, 2a+3b, 6a+11b, 22a+39b, \ldots >$$

$$< z_n > = < s_{2,n} > = < a, b, a^2b^3, a^6b^{11}, a^{22}b^{39}, \ldots >$$

$$s_{3,n+1} = \left(s_{3,n}^{s_{3,n}^{s_{3,n}}}\right)^{\left(s_{3,n-1}{}^{s_{3,n-1}}\right)} \quad \textit{and} \quad < s_{3,n} > = < a, b, b^{b^b.a^a}, \left(b^{b^b.a^a}\right)^{\left(\left((b^{b^b.a^a}\right)^{\left(b^{b^b.a^a}\right)}\right).b^b}, \ldots > .$$

5. Summary

A sequence of functions $< s_{1,n}, s_{2,n}, \ldots >$ has been defined, (with $s_{1,n}$ the Horadam function $w_n(a,b;p,-q)$ and $s_{2,n} = z_n$), each element of which is generated by a Horadam-like recurrence relation, with higher order operations than the previous one. The first two of these can be represented in terms of elementary arithmetical functions, z_n can also be written in terms of w_n. Later functions in the sequence, it seems, cannot be represented in terms of such elementary functions except for specific values of n. Perhaps later work, maybe with new notation, can change this situation.

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