ON A RESULT OF BUNDER INVOLVING HORADAM SEQUENCES: A NEW PROOF

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Abstract. This note offers a new proof of a 1975 result due to M. W. Bunder which has recently been proven (inductively), extended empirically and generalized in this journal. The proof methodology, while interesting, cannot be applied realistically beyond the original order two case of Bunder dealt with here.

1. Introduction

In a recent note [2] Larcombe and Bagdasar revisited a 1975 observation of M. W. Bunder which involves so called Horadam sequences. A simple inductive proof was given, his result extended empirically and a generalized version stated. Since two seminal and oft cited 1965 papers appeared by A. F. Horadam, this type of sequence has been studied continuously for almost half a century and a great many of its properties are known (see a survey article [3] by the authors for more details). Here we present an alternative first principles proof construction of Bunder's result which is new, being based on (degenerate and non-degenerate characteristic root) closed forms for the general term of a Horadam sequence.

Consider the Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(w_0, w_1; p, q)\}_0^{\infty}$ defined, for given w_0, w_1, by the order two linear recurrence

$$w_n = pw_{n-1} - qw_{n-2}, \qquad n \geq 2. \tag{1.1}$$

Bunder [1] noted that, given $z_0 = a$, $z_1 = b$, then defining a sequence $\{z_n(z_0, z_1, p, q)\}_0^{\infty}$ through the power product recurrence

$$z_n = (z_{n-1})^p (z_{n-2})^q, \qquad n \geq 2, \tag{1.2}$$

delivers (typeset with errors in [1])

$$\{z_n(a, b, p, q)\}_0^{\infty} = \{a, b, a^q b^p, a^{pq} b^{p^2+q}, a^{p^2q+q^2} b^{p^3+2pq}, a^{p^3q+2pq^2} b^{p^4+3p^2q+q^2}, \ldots\} \tag{1.3}$$

whose general $(n+1)$th term has, for $n \geq 0$, a closed form

$$z_n = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)} \tag{1.4}$$

featuring z_0, z_1 and, moreover, terms of two particular initial values instances $\{w_n(1, 0; p, -q)\}_0^{\infty}$ and $\{w_n(0, 1; p, -q)\}_0^{\infty}$ of Horadam sequences.

While the inductive proof in [2] is straightforward—and is readily applied, by natural extension, to the generalized version of the result stated therein—it offers, due to its very nature, little insight into Bunder's observation. It is, therefore, felt instructive to formulate an alternative proof by appealing directly to those closed forms of $w_n(w_0, w_1; p, q)$ required. The characteristic equation associated with (1.1) is

$$0 = \lambda^2 - p\lambda + q, \tag{1.5}$$

with roots $\alpha(p, q) = \frac{1}{2}(p + \sqrt{p^2 - 4q})$, $\beta(p, q) = \frac{1}{2}(p - \sqrt{p^2 - 4q})$, and in order to establish fully Bunder's result we must necessarily consider both the degenerate and non-degenerate characteristic root case solutions for w_n. If $a, b > 0$ then taking logs of the recurrence (1.2) gives $\ln(z_n) = p\ln(z_{n-1}) + q\ln(z_{n-2})$, and writing $t_n = \ln(z_n)$ defines a new Horadam sequence

$$\{t_n(a, b, p, q)\}_0^\infty = \{w_n(\ln(a), \ln(b); p, -q)\}_0^\infty \tag{1.6}$$

satisfying the recurrence $t_n = pt_{n-1} + qt_{n-2}$ $(n \geq 2)$ and accommodating the initial a, b values of $\{z_n\}_0^\infty$; it is closed forms for t_n, facilitated by those known for w_n, which provide an easy and obvious way to proceed.

2. The Proof

As just stated, there are two cases to prove separately. The closed forms for w_n which we utilize are built in standard fashion from characteristic roots with initial conditions imposed (we omit the details as they are trivial). For $p^2 \neq 4q$, and distinct (non-degenerate) characteristic roots $\alpha(p, q), \beta(p, q)$ as above, the so called Binet closed form for the Horadam sequence general term for $n \geq 0$ is

$$w_n(w_0, w_1; p, q) = \frac{(w_1 - w_0\beta)\alpha^n - (w_1 - w_0\alpha)\beta^n}{\alpha - \beta}. \tag{2.1}$$

For $p^2 = 4q$ on the other hand, and repeated (degenerate) characteristic roots $\alpha(p) = \beta(p) = \frac{1}{2}p$, the closed form for the Horadam sequence general term in this instance for $n \geq 0$ is

$$w_n(w_0, w_1; p, p^2/4) = w_1 n\alpha^{n-1} - w_0(n-1)\alpha^n. \tag{2.2}$$

As a point of interest we remark that a seemingly little known alternative route to them can be found in a 1960 text by Niven and Zuckerman [4, Section 4.4, pp. 90-92]; a clever rearrangement of a linear combination of Horadam sequence terms—which employs the basic recurrence equation (1.1) in conjunction with elementary properties of non-degenerate characteristic roots—establishes (2.1) in the first instance, followed by a limiting argument applied to this closed form which accounts for degenerate (equal) roots and duly gives (2.2).

Proof. (Case A: Non-Degeneracy.) It follows directly from (1.6) and (2.1) that, for $p^2 \neq -4q$, t_n has form

$$t_n = \frac{[\ln(b) - \ln(a)\hat{\beta}]\hat{\alpha}^n - [\ln(b) - \ln(a)\hat{\alpha}]\hat{\beta}^n}{\hat{\alpha} - \hat{\beta}}, \quad n \geq 0, \tag{2.3}$$

with distinct roots $\hat{\alpha}(p, q), \hat{\beta}(p, q), = \frac{1}{2}(p \pm \sqrt{p^2 + 4q})$ (from its governing characteristic equation $0 = \lambda^2 - p\lambda - q$). Rearranging, we write

$$t_n = A_1(\hat{\alpha}, \hat{\beta}, n)\ln(a) + A_2(\hat{\alpha}, \hat{\beta}, n)\ln(b) = \ln(z_n), \tag{2.4}$$

where

$$A_1(\hat{\alpha}, \hat{\beta}, n) = \frac{\hat{\alpha}\hat{\beta}^n - \hat{\beta}\hat{\alpha}^n}{\hat{\alpha} - \hat{\beta}}, \qquad A_2(\hat{\alpha}, \hat{\beta}, n) = \frac{\hat{\alpha}^n - \hat{\beta}^n}{\hat{\alpha} - \hat{\beta}}, \tag{2.5}$$

and so

$$z_n = a^{A_1(\hat{\alpha}, \hat{\beta}, n)} b^{A_2(\hat{\alpha}, \hat{\beta}, n)}. \tag{2.6}$$

Noting that by (2.1) $w_n(w_0, w_1; p, -q) = [(w_1 - w_0\hat{\beta})\hat{\alpha}^n - (w_1 - w_0\hat{\alpha})\hat{\beta}^n]/(\hat{\alpha} - \hat{\beta})$, the result follows since it is seen trivially that $w_n(1, 0; p, -q) = A_1(\hat{\alpha}, \hat{\beta}, n)$ and $w_n(0, 1; p, -q) = A_2(\hat{\alpha}, \hat{\beta}, n)$. □

Case A reveals how the power functions of (1.4) are manifested, and why they each have their particular form. While the precise details differ, the degenerate Case B is, of course, similar in nature.

Proof. (Case B: Degeneracy.) For $p^2 = -4q$ then by (2.2) t_n evidently has the corresponding form $(n \geq 0)$ $t_n = \ln(b)n\alpha^{n-1} - \ln(a)(n-1)\alpha^n = B_1(\alpha, n)\ln(a) + B_2(\alpha, n)\ln(b)$, say (constructed from the same double root $\alpha(p)$ (independent of q)), and, as required, we see that $B_1(\alpha, n) = (1-n)\alpha^n$ and $B_2(\alpha, n) = n\alpha^{n-1}$ coincide with (resp.) $w_n(1, 0; p, -q)$ and $w_n(0, 1; p, -q)$. ☐

Note that the restriction of positive a, b in writing (1.6) does not apply to Bunder's eventual result (1.4) which is valid for arbitrary starting values $a = z_0$, $b = z_1$.

3. Summary

In this note we have provided a new proof of a result, given many years ago by M. W. Bunder, which offers an insight that the previous inductive argument of [2] lacks. The method does not, however, realistically lend itself to the proof of any extended version of his observation beyond the order two case considered here, for that would require working from a characteristic equation of degree three or more. Closed form characteristic solutions are largely intractable or impossible to formulate/manipulate analytically when all $2k$ parameters of a general kth order linear recurrence equation (that is, recursion constants and initial values) are, for $k \geq 3$, held intact symbolically; for this reason, our proof here is a stand-alone one.

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References

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