ANOTHER PROBABILISTIC PROOF OF A BINOMIAL IDENTITY

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ABSTRACT. J. Peterson (2013) gave a simple and interesting proof of a binomial identity using exponential random variables. In this note, we give another elementary and short proof using uniformly distributed random variables.

Recently Peterson [2] gave a simple and interesting proof of the binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\theta}{\theta+k}=\prod_{k=1}^{n}\frac{k}{\theta+k} \quad \text{for } \theta>0, \quad n=1,2,\dots, \tag{1}$$

which also appeared in equation (5.41) in [1]. Several properties of exponential random variables were effectively used in his proof. In this note, we give another elementary and short proof using uniformly distributed random variables on $[0,1]$.

For $n\geq 1$ let $U_1,U_2,\dots,U_n$ be independent Unif$([0,1])$ random variables, where Unif$([0,1])$ denotes the uniform distribution on $[0,1]$. For $t\in[0,1]$ we then have

$$\mathrm{P}\left(\min_{1\leq i\leq n}\{U_i\}>t\right)=\mathrm{P}\left(\bigcap_{i=1}^{n}\{U_i>t\}\right)=(1-t)^n \tag{2}$$

$$=\sum_{k=0}^{n}\binom{n}{k}(-1)^k t^k. \tag{3}$$

Although the last equality follows from the binomial theorem, we note that (3) can be also directly verified by the inclusion-exclusion principle.

Now, let V be a Unif$([0,1])$ random variable which is independent of $\{U_i\}_{i=1}^n$. For $\theta>0$ it follows that

$$\begin{aligned}\mathrm{P}\left(\min_{1\leq i\leq n}\{U_i\}>V^{1/\theta}\right)&=\mathrm{E}\left[\mathrm{P}\left(\bigcap_{i=1}^{n}\{U_i>V^{1/\theta}\}\,\middle|\,V\right)\right]\\&=\int_0^1\mathrm{P}\left(\min_{1\leq i\leq n}\{U_i\}>x^{1/\theta}\right)dx.\end{aligned}$$

Applying (2) and (3) to this probability yields two different expressions. Equation (2) provides

$$\begin{aligned}\mathrm{P}\left(\min_{1\leq i\leq n}\{U_i\}>V^{1/\theta}\right)&=\int_0^1(1-x^{1/\theta})^n dx=\theta\int_0^1(1-t)^n t^{\theta-1}dt\\&=\theta\mathrm{Beta}(n+1,\theta)=\theta\frac{\Gamma(n+1)\Gamma(\theta)}{\Gamma(n+1+\theta)}\\&=\frac{\theta n!\Gamma(\theta)}{(n+\theta)(n-1+\theta)\cdots\theta\Gamma(\theta)}=\prod_{k=1}^{n}\frac{k}{\theta+k},\end{aligned}$$

where $\Gamma(\cdot)$ and $\text{Beta}(\cdot,\cdot)$ are standard gamma and beta functions, respectively. On the other hand, equation (3) provides

$$\mathrm{P}\left(\min_{1\le i\le n}\{U_i\} > V^{1/\theta}\right) = \int_0^1 \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^{k/\theta} dx = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \frac{\theta}{k+\theta}.$$

This completes the proof.

Remark. Letting $\text{Exp}(\lambda)$ be the exponential distribution with parameter $\lambda > 0$, namely the density is $\lambda e^{-\lambda x}$ for $x > 0$, we suppose that $X_1, \ldots, X_n$ are independent Exp(1) random variables, and T is an $\text{Exp}(\theta)$ random variable which is independent of X_i for all $i = 1, \ldots, n$. Note that the probability in this note $\mathrm{P}\left(\min_{1\le i\le n}\{U_i\} > V^{1/\theta}\right)$ is equivalent to $\mathrm{P}(\max_{1\le i\le n}\{X_i\} < T)$ which was studied by Peterson [2], because the distribution of $-\log(1-U)/\lambda$ is $\text{Exp}(\lambda)$, where U is a $\text{Unif}([0,1])$ random variable.

References

[1] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd. ed., Reading, MA, 1994.

[2] J. Peterson, *A Probabilistic Proof of a Binomial Identity*, American Math. Monthly, **120.6** (2013), 558–562.

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