

SUBSEQUENCES AND DIVISIBILITY BY POWERS OF THE FIBONACCI NUMBERS

KRITKHAJOHN ONPHAENG AND PRAPANPONG PONGSRIIAM

ABSTRACT. Let F_n be the n th Fibonacci number. Let m, n be positive integers. Define a sequence $(G(k, n, m))_{k \geq 1}$ by $G(1, n, m) = F_n^m$, and $G(k + 1, n, m) = F_n G(k, n, m)$ for all $k \geq 1$. We show that $F_n^{k+m-1} \mid G(k, n, m)$ for all $k, m, n \in \mathbb{N}$. Then we calculate $\frac{G(k, n, m)}{F_n^{k+m-1}} \pmod{F_n}$.

1. INTRODUCTION

The Fibonacci sequence is defined by the recurrence relation $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. These numbers are famous for possessing wonderful properties, see [1, 6, 7, 16] for additional references and history. The Fibonacci Association was formed in order to provide enthusiasts an opportunity to exchange ideas about Fibonacci numbers and related topics. In this article, we would like to share an idea on the divisibility properties of the sequence $(G(k, n, m))_{k \geq 1}$ defined as follows: for each $n, m \geq 1$, let

$$G(1, n, m) = F_n^m \text{ and let } G(k + 1, n, m) = F_n G(k, n, m) \text{ for all } k \geq 1.$$

For example, the first few terms of this sequence are $F_n^m, F(nF_n^m), F(nF(nF_n^m))$, where we write $F(\ell)$ instead of F_ℓ for convenience. So $(G(k, n, m))_{k \geq 2}$ (neglecting the term $k = 1$) is a subsequence of Fibonacci numbers. Our work is motivated by the results in [10, 15]. Recall that for integers $d \geq 2, k \geq 0$, and $a \geq 1$ we say that d^k exactly divides a and write $d^k \parallel a$ if $d^k \mid a$ and $d^{k+1} \nmid a$. Tangboonduangjit and Wiboonton [15] show that $F_n^k \mid G(k, n, 1)$ for all $n, k \in \mathbb{N}$. Then Panraksa, Tangboonduangjit, and Wiboonton [10] show that $F_n^k \parallel G(k, n, 1)$ for all $n \geq 4, k \geq 1$. They [10] also calculate $\frac{G(k, n, 1)}{F_n^k} \pmod{F_n}$ for $k = 2$ and $k = 3$. In this article, we extend their result [10, 15] by showing that $F_n^{k+m-1} \mid G(k, n, m)$ for every $k, m, n \in \mathbb{N}$. We also calculate $\frac{G(k, n, m)}{F_n^{k+m-1}} \pmod{F_n}$ for all $k \geq 2$ and $m \geq 1$. The precise statements are as follows.

Theorem 1.1. *The following statements hold:*

- (i) $F_n^{k+m-1} \mid G(k, n, m)$ for every $k, m, n \geq 1$.
- (ii) Let $k \geq 2$ and $n \geq 3$. Then

The second author received financial support from Faculty of Science, Silpakorn University, Thailand, contract number RGP 2555-07.

Correspondence should be addressed to Prapanpong Pongsriiam: prapanpong@gmail.com.

$$\frac{G(k, n, m)}{F_n^{k+m-1}} \equiv \begin{cases} 1, & \text{if } 2 \mid k \text{ and } 3 \nmid n \text{ or if } 2 \nmid k, 3 \nmid n, \text{ and } 4 \nmid n; \\ F_{n-1}, & \text{if } 2 \nmid k, 3 \nmid n \text{ and, } 4 \mid n; \\ \left(\frac{F_{n-3}}{2}\right)^{k-1}, & \text{if } 2 \nmid k, 3 \mid n, \text{ and } m \geq 2 \text{ or} \\ & \text{if } 2 \mid k, 3 \mid n \text{ and } m = 1; \\ (-1)^n \left(\frac{F_{n-3}}{2}\right)^{k-1}, & \text{if } 2 \mid k, 3 \mid n, \text{ and } m \geq 2 \text{ or} \\ & \text{if } 2 \nmid k, 3 \mid n, \text{ and } m = 1, \end{cases}$$

where the congruence is taken modulo F_n .

Substituting $k = 2, 3$ and $m = 1$ in Theorem 1.1, we obtain the results in [15]. With a bit more work, we obtain $F_n^{k+m-1} \parallel G(k, n, m)$ for every $k \geq 2, n \geq 4$, and $m \geq 1$, which also extends the results in [10, 15]. Let us record it in the next Corollary.

Corollary 1.2. *The following statements hold:*

- (i) $F_n^{k+m-1} \parallel G(k, n, m)$ for every $k \geq 2, n \geq 4, m \geq 1$.
- (ii) For $n \geq 3, \frac{G(2, n, 1)}{F_n^2} = \frac{F_n F_n}{F_n^2} \equiv \begin{cases} 1 \pmod{F_n}, & \text{if } 3 \nmid n; \\ \frac{1}{2} F_{n-3} \pmod{F_n}, & \text{if } 3 \mid n. \end{cases}$
- (iii) For $n \geq 3, \frac{G(3, n, 1)}{F_n^3} = \frac{F(n F_n F_n)}{F_n^3} \equiv \begin{cases} 1 \pmod{F_n}, & \text{if } 3 \nmid n \text{ and } 4 \nmid n; \\ F_{n-1} \pmod{F_n}, & \text{if } 3 \nmid n \text{ and } 4 \mid n; \\ (-1)^n \left(\frac{F_{n-3}}{2}\right)^2 \pmod{F_n}, & \text{if } 3 \mid n. \end{cases}$

To prove Theorem 1.1, we need a number of well-known identities listed in the next section.

2. PRELIMINARIES AND LEMMAS

Let m, n , and r be positive integers. The following results are well-known and will be used throughout this article :

$$\gcd(F_m, F_n) = F_{\gcd(m, n)}. \tag{2.1}$$

$$F_{nr} = \sum_{j=1}^r \binom{r}{j} F_n^j F_{n-1}^{r-j} F_j. \tag{2.2}$$

$$\text{If } m \geq 3, \text{ then } F_m \mid F_n \text{ if and only if } m \mid n. \tag{2.3}$$

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n. \tag{2.4}$$

$$F_{n-1}^2 \equiv F_{n+1}^2 \equiv (-1)^n \pmod{F_n}. \tag{2.5}$$

$$F_{m+n} = F_{m+1} F_n + F_m F_{n-1}. \tag{2.6}$$

$$\text{If } m \equiv n \pmod{6}, \text{ then } F_m \equiv F_n \pmod{4}. \tag{2.7}$$

For the reader's convenience, let us give references for the above identities. The relation (2.1) and (2.3) can be found in [7, p. 197-198], (2.4) is the Cassini's identity [7, p. 74], (2.5) follows from Cassini's identity and the fact that $F_{n+1} \equiv F_{n-1} \pmod{F_n}$. The identity (2.2) is an

important tool in proving our theorem and can be found in several articles such as [2, 4, 5]. The identity (2.6) is proved in [3, p. 294]. For the relation (2.7), noticing that $F_0 \equiv F_6$ and $F_1 \equiv F_7 \pmod{4}$, we see from the recursive definition of the Fibonacci sequence that $(F_n) \pmod{4}$ has period 6. Therefore (2.7) holds. The following is a consequence of (2.7) which will be used repeatedly.

$$\text{If } 3 \nmid n \text{ and } 2 \nmid n, \text{ then } F_n \equiv 1 \pmod{4}. \tag{2.8}$$

The proof of (2.8) is as follows. Assume that $3 \nmid n$ and $2 \nmid n$. Then $n \equiv 1, 5 \pmod{6}$. Therefore $F_n \equiv F_1, F_5 \pmod{4}$ by (2.7). Since $F_1 = 1 \equiv 1, F_5 = 5 \equiv 1 \pmod{4}$, we have $F_n \equiv 1 \pmod{4}$. We refer the reader to [1, 6, 7, 8, 12, 13, 16, 17] for more details and references. The next lemma is an important tool to prove our results.

Lemma 2.1. [2, 5] *For each positive integer n, r , and s , if $F_n^{s-1} \mid r$, then $F_n^s \mid F_{nr}$.*

Note that Lemma 2.1 was first proved in 1977 by V. E. Hoggatt, Jr., and Marjorie Bicknell-Johnson [5] but it or its consequence has been proved again recently [2, 9, 14] by a different method. It is worth noting that many authors [2, 9, 14] do not seem to realize the existence of Lemma 2.1 which was first proved in [5]. For example, since $F_n^k \mid F_n^k$, we obtain from Lemma 2.1 immediately that

$$F_n^{k+1} \mid F_{nF_n^k}. \tag{2.9}$$

It was mentioned by D. Marques [9, p. 241] that to the best of his knowledge, (2.9) was first proved by Benjamin and Rouse [2, 2006], using a combinatorial approach, and a second proof of (2.9) is due to Seibert and Trojovský [14, 2008] by using mathematical induction together with an identity for $\frac{F_{nm}}{F_m}$. D. Marques [9, 2012] himself also gave another proof of (2.9) by applying Lengyel’s Theorem [8] on the p -adic order of Fibonacci and Lucas numbers. So it is nice to bring Lemma 2.1 back to the literature.

The following lemmas are of our own. We also prove Lemma 2.2 and Lemma 2.3 in [11] but our manuscript has not been formally published in a journal. So we give a proof here for completeness.

Lemma 2.2. *Let $a, j \geq 1$ and $s \geq 2$ be positive integers. Assume that c is the smallest nonnegative integer such that $j \mid as^c$. Then there exists a prime p such that $p \mid s$ and $p^c \mid j$.*

Proof. If $c = 0$, then we can choose any prime p that divides s . So we assume that $c \geq 1$. Since $j \mid as^c$, there is $q \in \mathbb{N}$ such that $as^c = jq$. If $s \mid q$, then $as^{c-1} = j\left(\frac{q}{s}\right)$ and therefore $j \mid as^{c-1}$ which contradicts the minimality of c . Thus, $s \nmid q$. Then there exists a prime p such that $p^\alpha \parallel s, p^\beta \parallel q$, and $\alpha > \beta \geq 0$. Since $q \mid as^c$, we see that

$$\frac{q}{p^\beta} \mid a \left(\frac{s}{p^\alpha}\right)^c. \tag{2.10}$$

Let $n = \frac{a\left(\frac{s}{p^\alpha}\right)^c}{\frac{q}{p^\beta}} \cdot p^{(\alpha-1)c-\beta}$. Then $p^c n = \frac{as^c}{q} = j$. Now by (2.10) and the fact that $\alpha > \beta$, we see that $n \in \mathbb{N}, p^c \mid j$, and $p \mid s$. □

Lemma 2.3. *Let $k, \ell, r, s \in \mathbb{N}$ and $s^k \mid r$. Then $s^{k+\ell} \mid \binom{r}{j} s^j$ for all $1 \leq j \leq r$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+2} \mid \binom{r}{j} s^j$ for all $3 \leq j \leq r$.*

Proof. Note that the result holds trivially when $s = 1$ or $j \geq k + \ell$. So we assume that $s \geq 2$ and $1 \leq j < k + \ell$ satisfying $2^{j-\ell+1} > j$. Since $s^k \mid r$, there is $b \in \mathbb{N}$ such that $r = s^k b$. Then

$$\binom{r}{j} = \frac{r}{j} \binom{r-1}{j-1} = \frac{s^k b}{j} \binom{r-1}{j-1} = s^{k-j+\ell} \cdot \frac{s^{j-\ell} b \binom{r-1}{j-1}}{j}.$$

Suppose for a contradiction that $j \nmid s^{j-\ell} b \binom{r-1}{j-1}$. Since $j \mid s^k b \binom{r-1}{j-1}$, there exists the smallest nonnegative integer c such that $j \mid s^c b \binom{r-1}{k-1}$. Then

$$j - \ell + 1 \leq c \leq k. \tag{2.11}$$

By Lemma 2.2, there exists a prime p such that $p^c \mid j$. Now $j \geq p^c \geq 2^c \geq 2^{j-\ell+1} > j$, a contradiction. Hence, $j \mid s^{j-\ell} b \binom{r-1}{j-1}$ and therefore $s^{k-j+\ell} \mid \binom{r}{j}$ as required. \square

Lemma 2.4. *Let $m, n \geq 1$, $k \geq 2$, and $r = G(k, n, m)$. Then the following statements hold.*

- (i) $2 \mid r$ if and only if $3 \mid n$ or $4 \mid n$.
- (ii) If $2 \nmid n$ and $3 \nmid n$, then $r \equiv 1 \pmod{4}$.
- (iii) If $3 \mid n$, then $r \equiv 0 \pmod{8}$.

Proof. Throughout the proof, we will apply (2.3) repeatedly for the case $m = 3, 4$ so let us record it here again.

$$3 \mid n \text{ if and only if } 2 \mid F_n \text{ and } 4 \mid n \text{ if and only if } 3 \mid F_n.$$

We will prove (i) by induction on k . Consider the equivalence:

$$\begin{aligned} 2 \mid G(2, n, m) &\Leftrightarrow F_3 \mid F_n F_n^m \Leftrightarrow 3 \mid n F_n^m \Leftrightarrow 3 \mid n \vee 3 \mid F_n \\ &\Leftrightarrow 3 \mid n \vee F_4 \mid F_n \Leftrightarrow 3 \mid n \vee 4 \mid n. \end{aligned}$$

This proves (i) when $k = 2$. Next, we consider the case $k = 3$.

$$\begin{aligned} 2 \mid G(3, n, m) &\Leftrightarrow 3 \mid n G(2, n, m) \Leftrightarrow 3 \mid n \vee 3 \mid G(2, n, m) \\ &\Leftrightarrow 3 \mid n \vee 4 \mid n F_n^m. \end{aligned} \tag{2.12}$$

If $4 \mid n F_n^m$ and $3 \nmid n$, then $(2, F_n) = 1$ which implies that $4 \mid n$. So (2.12) is equivalent to the condition $3 \mid n$ or $4 \mid n$. This proves (i) for $k = 3$. Now assume that $k \geq 3$ and (i) holds for $2, 3, \dots, k$. Then

$$\begin{aligned} 2 \mid G(k+1, n, m) &\Leftrightarrow 3 \mid n G(k, n, m) \\ &\Leftrightarrow 3 \mid n \vee 3 \mid G(k, n, m) \\ &\Leftrightarrow 3 \mid n \vee 4 \mid n G(k-1, n, m). \end{aligned} \tag{2.13}$$

If $2 \mid G(k-1, n, m)$, we can apply the induction hypothesis to conclude that $3 \mid n$ or $4 \mid n$. If $2 \nmid G(k-1, n, m)$, then $\gcd(4, G(k-1, n, m)) = 1$, and therefore $4 \mid n$. So (2.13) is equivalent to $3 \mid n$ or $4 \mid n$. This shows that (i) holds for all $k \geq 2$.

For (ii), we assume that $2 \nmid n$ and $3 \nmid n$. Again we will prove that $r \equiv 1 \pmod{4}$ by induction on k . Similar to the proof of (i), we see that $3 \mid n F_n^m \Leftrightarrow 3 \mid n$ or $4 \mid n$ and $2 \mid n F_n^m \Leftrightarrow 2 \mid n$ or $3 \mid n$. Since $2 \nmid n$ and $3 \nmid n$, we see that $3 \nmid n F_n^m$ and $2 \nmid n F_n^m$. By (2.8), $G(2, n, m) = F_n F_n^m \equiv 1 \pmod{4}$. The case $k = 3$ can be obtained similarly. Next assume that $k \geq 3$ and (ii) holds for $2, 3, \dots, k$. Then

$$\begin{aligned} 3 \mid n G(k, n, m) &\Leftrightarrow 3 \mid n \vee 4 \mid n G(k-1, n, m) \\ &\Leftrightarrow 3 \mid n \vee 4 \mid n \end{aligned} \tag{2.14}$$

where the penultimate equivalence is done by applying (i). From (2.14) and the fact that $3 \nmid n$ and $2 \nmid n$, we see that $3 \nmid nG(k, n, m)$. We also have by the induction hypothesis that $2 \nmid G(k, n, m)$ and therefore $2 \nmid nG(k, n, m)$. Since $3 \nmid nG(k, n, m)$ and $2 \nmid nG(k, n, m)$, we obtain by (2.8) that $G(k+1, n, m) = F_{nG(k, n, m)} \equiv 1 \pmod{4}$. This proves (ii). The proof of (iii) is similar. Hence the proof is complete. \square

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1(i). Since $G(1, n, m) = F_n^m$, the result holds for $k = 1$. If $k \geq 1$ and $F_n^{k+m-1} \mid G(k, n, m)$, then we obtain by Lemma 2.1 that $F_n^{k+1+m-1}$ divides $F_{nG(k, n, m)} = G(k+1, n, m)$. Therefore the result follows by induction on k . \square

Proof of Theorem 1.1(ii).

Case $m = 1$. Basis Step: We let $m = 1$ and we will prove the statement by induction on k . The case $k = 2$ is proved in [10]. But we will prove it again here. Our proof is shorter and gives an idea on the proof of the induction step and the proof of the case $m \geq 2$. The statement we want to prove is as follows:

$$\frac{F_n F_{F_n}}{F_n^2} \equiv \begin{cases} 1 \pmod{F_n}, & \text{if } 3 \nmid n; \\ \frac{1}{2} F_{n-3} \pmod{F_n}, & \text{if } 3 \mid n. \end{cases}$$

Let $r = F_n$, $A = F_{n-1}^{r-1}$, and $B = \frac{r(r-1)}{2} F_{n-1}^{r-2}$. Since $F_n \mid r$, we obtain by Lemma 2.3 that $F_n^3 \mid \binom{r}{j} F_n^j$ for all $3 \leq j \leq r$. That is $F_n \mid \binom{r}{j} F_n^{j-2}$ for each $3 \leq j \leq r$. Then by (2.2), we have

$$\frac{F_{nr}}{F_n^2} = \sum_{j=1}^r \binom{r}{j} F_n^{j-2} F_{n-1}^{r-j} F_j \equiv \sum_{j=1}^2 \binom{r}{j} F_n^{j-2} F_{n-1}^{r-j} F_j \equiv A + B \pmod{F_n}.$$

Case 1. Assume that $3 \nmid n$. Then $n \equiv 1, 2, 4, 5 \pmod{6}$. If $n \equiv 1, 2, 5$, then $F_n \equiv F_1, F_2, F_5 \equiv 1 \pmod{4}$, by (2.7). Therefore F_n is odd and $\frac{r-1}{2} = \frac{F_n-1}{2}$ is even. This implies that

$$B = \left(\frac{r-1}{2} \right) F_n F_{n-1}^{r-2} \equiv 0 \pmod{F_n},$$

and

$$A = F_{n-1}^{r-1} = F_{n-1}^{2\left(\frac{r-1}{2}\right)} \equiv (-1)^{n\left(\frac{r-1}{2}\right)} \equiv 1 \pmod{F_n}, \text{ by (2.5).}$$

Hence, $\frac{F_{nr}}{F_n^2} \equiv A + B \equiv 1 \pmod{F_n}$. In what follows, (2.5) and (2.7) may be applied without mentioning. Case 2. Assume that $3 \mid n$. Then $n \equiv 0, 3 \pmod{6}$. If $n \equiv 0 \pmod{6}$, then n is even, $F_n \equiv F_0 \equiv 0 \pmod{4}$, and therefore

$$F_{n-1}^{r-2} \equiv (-1)^{n\left(\frac{r-2}{2}\right)} \equiv 1 \pmod{F_n}.$$

Similarly, if $n \equiv 3 \pmod{6}$, then $F_n \equiv F_3 \equiv 2 \pmod{4}$ so $\frac{r-2}{2} = \frac{F_n-2}{2}$ is even and $F_{n-1}^{r-2} \equiv 1 \pmod{F_n}$. In any case, we have $F_{n-1}^{r-2} \equiv 1 \pmod{F_n}$. Therefore

$$\begin{aligned} \frac{F_{nr}}{F_n^2} &\equiv A + B = F_{n-1}^{r-2} \left(F_{n-1} - \frac{1}{2} F_n + \frac{1}{2} F_n^2 \right) \\ &= F_{n-1}^{r-2} \left(\frac{1}{2} F_{n-3} + \frac{1}{2} F_n^2 \right) \equiv \frac{1}{2} F_{n-3} \pmod{F_n}. \end{aligned}$$

This completes the proof of the basis step $k = 2$.

Induction Step: Next, let $k \geq 2$ and assume that the statement is true for k . Let $r = G(k, n, 1)$. Consider

$$\frac{G(k+1, n, 1)}{F_n^{k+1}} = \frac{F_{nr}}{F_n^{k+1}} = \sum_{j=1}^r \binom{r}{j} F_n^{j-k-1} F_{n-1}^{r-j} F_j.$$

Since $F_n^k \mid r$, we obtain by Lemma 2.3 that $F_n^{k+2} \mid \binom{r}{j} F_n^j$ for each $3 \leq j \leq r$. Therefore, $F_n \mid \binom{r}{j} F_n^{j-k-1}$ for each $3 \leq j \leq r$. So the above sum is congruent to $A + B$ modulo F_n where $A = rF_n^{-k}F_{n-1}^{r-1}$, $B = \binom{r}{2}F_n^{-k+1}F_{n-1}^{r-2}$. We have 6 cases to consider.

Case 1. Assume that $2 \nmid k$, $3 \nmid n$, and $4 \nmid n$. By Lemma 2.4(i), r is odd. Therefore,

$$B = \left(\frac{r-1}{2}\right) \frac{r}{F_n^k} F_n F_{n-1}^{r-2} \equiv 0 \pmod{F_n}.$$

By the induction hypothesis, $\frac{r}{F_n^k} \equiv 1 \pmod{F_n}$. Therefore,

$$A \equiv F_{n-1}^{r-1} \equiv (-1)^{n\binom{r-1}{2}} \pmod{F_n}.$$

If $2 \mid n$, then $A \equiv 1 \pmod{F_n}$. If $2 \nmid n$, then by Lemma 2.4(ii), $r \equiv 1 \pmod{4}$, which implies that $A \equiv 1 \pmod{F_n}$. In any case, $A \equiv 1 \pmod{F_n}$ and hence $A + B \equiv 1 \pmod{F_n}$.

Case 2. Assume that $2 \mid k$, $3 \nmid n$, and $4 \nmid n$. Then r is odd, by Lemma 2.4(i). The calculation of $A + B \pmod{F_n}$ is the same as that in Case 1 and we obtain $A + B \equiv 1 \pmod{F_n}$.

Case 3. Suppose that $2 \nmid k$, $3 \nmid n$, and $4 \mid n$. Then r is even by Lemma 2.4(i). We write $2(A + B) = \frac{r}{F_n^k} F_{n-1}^{r-2} (2F_{n-1} + (r-1)F_n)$. By the induction hypothesis,

$$\frac{r}{F_n^k} \equiv F_{n-1} \pmod{F_n}.$$

Since r and n are even, we obtain $F_{n-1}^{r-2} \equiv (-1)^{n\binom{r-2}{2}} \equiv 1 \pmod{F_n}$. Therefore,

$$2(A + B) \equiv F_{n-1} (2F_{n-1}) \equiv 2F_{n-1}^2 \equiv 2(-1)^n \equiv 2 \pmod{F_n}.$$

Since $3 \nmid n$, we see that $(2, F_n) = 1$ and thus $A + B \equiv 1 \pmod{F_n}$.

Case 4. Suppose that $2 \mid k$, $3 \nmid n$, and $4 \mid n$. Then by Lemma 2.4(i), r is even. The calculation of $A + B \pmod{F_n}$ is similar to the one in Case 3 except that, in this case, $\frac{r}{F_n^k} \equiv 1 \pmod{F_n}$.

We obtain $A + B \equiv F_{n-1} \pmod{F_n}$.

Case 5. Assume that $2 \nmid k$ and $3 \mid n$. Then F_n is even and, by Lemma 2.4(iii), $r \equiv 0 \pmod{8}$. We write

$$A + B = \frac{r}{F_n^k} F_{n-1}^{r-2} \left(F_{n-1} + \frac{r-1}{2} F_n \right).$$

We have $\frac{r}{F_n^k} \equiv (-1)^n \left(\frac{F_{n-3}}{2}\right)^{k-1} \pmod{F_n}$, by the induction hypothesis. In addition $F_{n-1}^{r-2} \equiv (-1)^{n\binom{r-2}{2}} \equiv (-1)^n \pmod{F_n}$ and $F_{n-1} + \frac{r-1}{2} F_n = F_{n-1} - \frac{F_n}{2} + \frac{rF_n}{2} = \frac{1}{2}F_{n-3} + \frac{rF_n}{2} \equiv \frac{1}{2}F_{n-3} \pmod{F_n}$. Therefore, $A + B \equiv \left(\frac{F_{n-3}}{2}\right)^k \pmod{F_n}$.

Case 6. Suppose that $2 \mid k$ and $3 \mid n$. Then F_n is even and, by Lemma 2.4(iii), $r \equiv 0 \pmod{8}$. Similar to Case 5, we obtain the following

$$\begin{aligned} \frac{r}{F_n^k} &\equiv \left(\frac{F_{n-3}}{2}\right)^{k-1}, \\ F_{n-1}^{r-2} &\equiv (-1)^{n\left(\frac{r-2}{2}\right)} \equiv (-1)^n, \quad \text{and} \\ F_{n-1} + \frac{r-1}{2}F_n &\equiv \frac{1}{2}F_{n-3}, \end{aligned}$$

where all congruences are taken modulo F_n . Therefore $A + B \equiv (-1)^n \left(\frac{F_{n-3}}{2}\right)^k \pmod{F_n}$.

Combining the results in Case 1 to Case 6, we obtain

$$\frac{G(k+1, n, 1)}{F_n^{k+1}} \equiv \begin{cases} 1, & \text{if } 2 \nmid k \text{ and } 3 \nmid n \text{ or if } 2 \mid k \text{ and} \\ & 3 \nmid n, 4 \nmid n \text{ (by Case 1, Case 2, and Case 3),} \\ F_{n-1}, & \text{if } 2 \mid k \text{ and } 3 \nmid n, 4 \mid n \text{ (by Case 4),} \\ \left(\frac{F_{n-3}}{2}\right)^k, & \text{if } 2 \nmid k \text{ and } 3 \mid n \text{ (by Case 5),} \\ (-1)^n \left(\frac{F_{n-3}}{2}\right)^k, & \text{if } 2 \mid k \text{ and } 3 \mid n \text{ (by Case 6).} \end{cases}$$

Since $2 \mid k$ if and only if $2 \nmid k+1$, we can write the above congruence in the following form:

$$\frac{G(k+1, n, 1)}{F_n^{k+1}} \equiv \begin{cases} 1, & \text{if } 2 \mid k+1 \text{ and } 3 \nmid n \text{ or if } 2 \nmid k+1 \text{ and} \\ & 3 \nmid n, 4 \nmid n, \\ F_{n-1}, & \text{if } 2 \nmid k+1 \text{ and } 3 \nmid n, 4 \mid n, \\ \left(\frac{F_{n-3}}{2}\right)^k, & \text{if } 2 \mid k+1 \text{ and } 3 \mid n, \\ (-1)^n \left(\frac{F_{n-3}}{2}\right)^k, & \text{if } 2 \nmid k+1 \text{ and } 3 \mid n, \end{cases}$$

where the congruences are taken modulo F_n . This proves the induction step and hence the proof of the case $m = 1$ and $k \geq 2$ is complete.

Case $m \geq 2$. Basis Step: We let $m \geq 2$ and we will prove this statement by induction on k . First we will consider the case $k = 2$. Similar to the proof of the Case $m = 1$, we let $r = F_n^m$ and we obtain that $\frac{G(2, n, m)}{F_n^{m+1}}$ is congruent to $A + B$ modulo F_n , where $A = F_{n-1}^{r-1}$ and $B = \frac{1}{2}F_n(r-1)F_{n-1}^{r-2}$. If $3 \nmid n$, then we can follow the same argument of the proof of Case $m = 1$, and we obtain $A + B \equiv 1 \pmod{F_n}$. Now assume that $3 \mid n$. The process of calculation is still the same as the proof of the Case $m = 1$. Only at this time, $m \geq 2$, so $F_{n-1}^{r-2} \equiv (-1)^n \pmod{F_n}$. Therefore,

$$A + B = F_{n-1}^{r-2} \left(\frac{1}{2}F_{n-3} + \frac{1}{2}rF_n \right) \equiv \frac{(-1)^n}{2}F_{n-3} \pmod{F_n}.$$

In conclusion,

$$\frac{G(2, n, m)}{F_n^{1+m}} \equiv \begin{cases} 1 \pmod{F_n}, & \text{if } 3 \nmid n; \\ (-1)^n \left(\frac{F_{n-3}}{2}\right) \pmod{F_n}, & \text{if } 3 \mid n. \end{cases}$$

This proves the basis step.

Induction Step: Next let $k \geq 2$ and assume that the statement is true for k . Let $r = G(k, n, m)$.

Then following the same argument as in proof of the Case $m = 1$, we see that $\frac{G(k+1,n,m)}{F_n^{k+m}}$ is congruent to $A + B$ modulo F_n , where $A = rF_n^{1-k-m}F_{n-1}^{r-1}$ and $B = \binom{r}{2}F_n^{-k-m+2}F_{n-1}^{r-2}$.

Case 1. If $2 \mid k$, $3 \nmid n$, and $4 \nmid n$ or $2 \nmid k$, $3 \nmid n$, and $4 \nmid n$, then we can follow the argument of Case 1 and Case 2 in the proof of the Case $m = 1$. We obtain $A + B \equiv 1 \pmod{F_n}$.

Case 2. If $2 \nmid k$, $3 \nmid n$, and $4 \mid n$ or $2 \mid k$, $3 \nmid n$, and $4 \mid n$, then the proof is similar to Case 3 in the proof of the Case $m = 1$. We obtain

$$A + B \equiv \begin{cases} 1 \pmod{F_n}, & \text{if } 2 \nmid k, 3 \nmid n, \text{ and } 4 \mid n; \\ F_{n-1} \pmod{F_n}, & \text{if } 2 \mid k, 3 \nmid n, \text{ and } 4 \mid n. \end{cases}$$

Case 3. If $2 \mid k$ and $3 \mid n$, then the proof is similar to Case 5 and we obtain $A + B \equiv \left(\frac{F_{n-3}}{2}\right)^k \pmod{F_n}$. If $2 \nmid k$ and $3 \mid n$, then the proof is similar to Case 6 and we have $A + B \equiv (-1)^n \left(\frac{F_{n-3}}{2}\right)^k \pmod{F_n}$.

This completes the proof. □

Proof of Corollary 1.2. Let $k \geq 2$, $n \geq 4$, and $m \geq 1$ be positive integers. Since $1 \not\equiv 0 \pmod{F_n}$ and $F_{n-1} \not\equiv 0 \pmod{F_n}$, we see that $F_n^{k+m-1} \parallel G(k, n, m)$ for every n such that $3 \nmid n$. So we assume that $3 \mid n$. Suppose for a contradiction that

$$F_n \mid \left(\frac{F_{n-3}}{2}\right)^{k-1}. \tag{3.1}$$

Since $\gcd(F_n, F_{n-3}) = F_{\gcd(n, n-3)} = F_3 = 2$, we obtain $\gcd\left(\frac{F_n}{2}, \frac{F_{n-3}}{2}\right) = 1$.

Then $\gcd\left(\frac{F_n}{2}, \left(\frac{F_{n-3}}{2}\right)^{k-1}\right) = 1$. But from (3.1), we have $\frac{F_n}{2} \mid \left(\frac{F_{n-3}}{2}\right)^{k-1}$. Therefore,

$$1 = \gcd\left(\frac{F_n}{2}, \left(\frac{F_{n-3}}{2}\right)^{k-1}\right) = \frac{F_n}{2},$$

which implies $n = 3$. This contradicts the fact that $n \geq 4$. Hence, $F_n \nmid \left(\frac{F_{n-3}}{2}\right)^{k-1}$ and thus $F_n^{k+m-1} \parallel G(k, n, m)$. This proves (i). By substituting $k = 2$ and $k = 3$ in Theorem 1.1, we immediately obtain (ii) and (iii). This completes the proof. □

4. ACKNOWLEDGEMENTS

We would like to thank C. Panraksa, A. Tangboonduangjit, and K. Wiboonton for giving us their documents [10, 15] which lead to the publication of this article. We also wish to thank the anonymous referee for his or her suggestions which improved the presentation of this article.

REFERENCES

- [1] A. Benjamin and J. Quinn, *The Fibonacci numbers—exposed more discretely*, Math. Mag., **76.3** (2003), 182–192.
- [2] A. Benjamin and J. Rouse, *When does F_m^L divide F_n ? A combinatorial solution*, Applications of Fibonacci Numbers, **10** (2006), 27–34.
- [3] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Second edition, Addison-Wesley (1994).
- [4] J. H. Halton, *On the divisibility properties of Fibonacci numbers*, The Fibonacci Quarterly, **4.3** (1966), 217–240.

SUBSEQUENCES AND DIVISIBILITY BY POWERS OF THE FIBONACCI NUMBERS

- [5] V. E. Hoggatt, Jr. and Marjorie Bicknell-Johnson, *Divisibility by Fibonacci and Lucas squares*, The Fibonacci Quarterly, **15.1** (1977), 3–8.
- [6] D. Kalman and R. Mena, *The Fibonacci numbers—exposed*, Math. Mag., **76.3** (2003), 167–181.
- [7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley (2001).
- [8] T. Lengyel, *The order of the Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **33.3** (1995), 234–239.
- [9] D. Marques, *The order of appearance of power of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **50.3** (2012), 239–245.
- [10] C. Panraksa, A. Tangboonduangjit, and K. Wiboonton, *Exact divisibility properties of some subsequences of Fibonacci numbers*, The Fibonacci Quarterly, **51.4** (2013), 307–318.
- [11] P. Pongsriam, *Exact divisibility by Fibonacci and Lucas powers*, preprint.
- [12] P. Ribenboim, *My Numbers, My Friends: Popular Lectures on Number Theory*, Springer–Verlag, New York, 2000.
- [13] D. W. Robinson, *The Fibonacci matrix modulo m* , The Fibonacci Quarterly, **1.1** (1963), 29–36.
- [14] J. Seibert and P. Trojovský, *On divisibility of a relation of the Fibonacci numbers*, Int. J. Pure Appl. Math., **46.3** (2008), 443–448.
- [15] A. Tangboonduangjit and K. Wiboonton, *Divisibility properties of some subsequences of Fibonacci numbers*, East–West J. Math., Spec. Vol., (2012), pp. 331–336.
- [16] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications*, Dover Publications, 2007.
- [17] J. Vinson, *The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence*, The Fibonacci Quarterly, **1.1** (1963), 37–45.

MSC2010: 11B39

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, PRINCE OF SONGKLA UNIVERSITY,
SONGKLA, 91000, THAILAND

E-mail address: dome3579@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SILPAKORN UNIVERSITY, NAKHON PATHOM, 73000,
THAILAND

E-mail address: prapanpong@gmail.com