p-ADIC STIRLING NUMBERS OF THE SECOND KIND

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Abstract. Let $S(n,k)$ denote the Stirling numbers of the second kind. We prove that the p-adic limit of $S(p^e a+c, p^e b+d)$ as $e\to\infty$ exists for any integers a, b, c, and d with $0<b\le a$. We call the limiting p-adic integer $S(p^\infty a+c, p^\infty b+d)$. When $a\equiv b \bmod (p-1)$ or $d\le 0$, we express them in terms of p-adic binomial coefficients $\binom{p^\infty \alpha-1}{p^\infty\beta}$ introduced in a recent paper.

1. Main Theorems

In [4], the author defined, for integers a, b, c, and d, with $0<b\le a$, $\binom{p^\infty a+c}{p^\infty b+d}$ to be the p-adic integer which is the p-adic limit of $\binom{p^e a+c}{p^e b+d}$, and gave explicit formulas for these in terms of rational numbers and p-adic integers which, if p or n is even, could be considered to be $\mathrm{U}_p((p^\infty n)!) := \lim_e \mathrm{U}_p((p^e n)!)$. Here and throughout, $\nu_p(-)$ denotes the exponent of p in an integer or rational number and $\mathrm{U}_p(n)=n/p^{\nu_p(n)}$ denotes the unit factor in n. Here we do the same for Stirling numbers $S(n,k)$ of the second kind; i.e., we prove that the p-adic limit of $S(p^e a+c, p^e b+d)$ exists if $0<b\le a$, and call it $S(p^\infty a+c, p^\infty b+d)$. If $a\equiv b \bmod (p-1)$ or $d\le 0$, we express these explicitly in terms of certain $\binom{p^\infty\alpha-1}{p^\infty\beta}$ together with certain Stirling-like rational numbers.

For nonnegative integers n and k, the Stirling number $S(n,k)$ of the second kind is the number of ways to partition a set of n objects into k nonempty subsets. (See, e.g., [2, p. 204].) The formula which is most useful to us is

$$S(n,k)=\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

Our interest in them was initially due to an occurrence in algebraic topology, related to homotopy groups of the special unitary groups [5].

We now list our four main theorems, which will be proved in Sections 2 and 4. Let $\mathbb{Z}_p$ denote the p-adic integers with the usual metric.

Theorem 1.1. *Let p be a prime, and a, b, c, and d integers with $0<a\le b$. Then the p-adic limit of $S(p^e a+c, p^e b+d)$ exists in $\mathbb{Z}_p$. We denote the limit as $S(p^\infty a+c, p^\infty b+d)$.*

It will be assumed throughout that $0<b\le a$. Note that if c and/or d are negative, then for some small values of e, $S(p^e a+c, p^e b+d)$ might have a negative argument, and hence not be defined. However, the p-adic limit only cares about large values of e, and for sufficiently large e, both arguments of $S(p^e a+c, p^e b+d)$ will be positive.

Theorem 1.2. *If p is any prime and $0<b\le a$, then $S(p^\infty a, p^\infty b)=0$ if $a\not\equiv b \bmod (p-1)$, while*

$$S(p^\infty a, p^\infty b)=\binom{p^\infty\frac{pa-b}{p-1}-1}{p^\infty\frac{p(a-b)}{p-1}} \text{ if } a\equiv b \bmod (p-1).$$

These p-adic binomial coefficients are as introduced in [4].

Let $|s(n,k)|$ denote the unsigned Stirling numbers of the first kind.

Theorem 1.3. *If* $0 < b \le a$, *then*

$$S(p^\infty a + c, p^\infty b + d) = \begin{cases} 0, & d = 0,\ c \ne 0; \\ 0, & d < 0,\ c \ge 0; \\ |s(|d|,|c|)| S(p^\infty a, p^\infty b), & c < 0,\ d < 0. \end{cases}$$

In particular, if $a \not\equiv b \bmod (p-1)$, then $S(p^\infty a + c, p^\infty b + d) = 0$ whenever $d \le 0$.

For any prime number p, integer n, and nonnegative integer k, define the partial Stirling numbers $T_p(n,k)$ [3] by

$$T_p(n,k) = \frac{(-1)^k}{k!} \sum_{i \not\equiv 0\ (p)} (-1)^i \binom{k}{i} i^n. \tag{1.1}$$

Theorem 1.4. *If* $a \equiv b \bmod (p-1)$ *and* $d \ge 1$, *then*

$$S(p^\infty a + d - 1, p^\infty b + d) = T_p(d-1,d) \binom{p^\infty \frac{pa-b}{p-1} - 1}{p^\infty b}.$$

When $a \equiv b \bmod (p-1)$, results for all $S(p^\infty a + c, p^\infty b + d)$ with $d > 0$ follow from these results and the standard formula

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \tag{1.2}$$

Explicit formulas are somewhat complicated and are relegated to Section 3.

We thank the referee for pointing out an oversight in an earlier version of the paper.

2. Proofs When $a \equiv b \bmod (p-1)$ or $d \le 0$

In this section, we prove Theorems 1.2, 1.3, and 1.4. If $a \equiv b \bmod (p-1)$ or $d \le 0$, Theorem 1.1 follows immediately from Theorems 1.2, 1.3, and 1.4 and their proofs. These give explicit values for the limits when $d \le 0$ and for at least one value of c when $d > 0$. The existence of the limit for other values of c follows from (1.2) and induction. Examples are given in Section 3. We will prove Theorem 1.1 when $a \not\equiv b \bmod (p-1)$ and $d > 0$ in Section 4.

We rely heavily on the following two results of Chan and Manna.

Theorem 2.1. ([1, 4.2, 5.2]) *Suppose* $n > p^m b$ *with* $m \ge 3$ *if* $p = 2$. *Then,* $\bmod p^{m-1}$ *if* $p = 2$, *and* $\bmod p^m$ *if* p *is odd,*

$$S(n, p^m b) \equiv \begin{cases} \binom{n/2 - 2^{m-2}b - 1}{n/2 - 2^{m-1}b}, & \text{if } p = 2 \text{ and } n \equiv 0 \bmod 2; \\ \binom{(n - p^{m-1}b)/(p-1) - 1}{(n - p^m b)/(p-1)}, & \text{if } p \text{ is odd and } n \equiv b \bmod (p-1); \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.2. ([1, 4.3, 5.3]) *Let* p *be any prime, and suppose* $n \ge p^e b + d$. *Then*

$$S(n, p^e b + d) \equiv \sum_{j \ge 0} S(p^e b + (p-1)j, p^e b) S(n - p^e b - (p-1)j, d) \bmod p^e.$$

Proof of Theorem 1.2. The result follows from Theorem 2.1. If p is odd and $a \not\equiv b \bmod (p-1)$, then $\nu_p(S(p^e a, p^e b)) \geq e$, while if $a \equiv b \bmod (p-1)$, then

$$S(p^e a, p^e b) \equiv \binom{p^{e-1}\frac{pa-b}{p-1} - 1}{p^{e-1}\frac{p(a-b)}{p-1}} \bmod p^e.$$

If $p = 2$, then

$$S(2^e a, 2^e b) \equiv \binom{2^{e-2}(2a-b) - 1}{2^{e-2}(2a-2b)} \bmod 2^{e-1}.$$

□

Let $d_p(n)$ denote the sum of the digits in the p-ary expansion of a positive integer n.

Proof of Theorem 1.3. The first case follows readily from Theorem 2.1. If $p = 2$, this says that $\nu(S(2^e a + c, 2^e b)) \geq e - 1$ if c is odd, while if $c = 2k$ is even, then, $\bmod 2^{e-1}$,

$$S(2^e a + 2k, 2^e b) \equiv \binom{2^{e-1}a + k - 2^{e-2}b - 1}{2^{e-1}a + k - 2^{e-1}b}.$$

If $0 < k < 2^{e-1}$, this has 2-exponent

$$\nu_2 = d_2(a-b) + d_2(k) - (d_2(2a-b) + d_2(k-1)) + d_2(2^{e-2}b - 1) \to \infty$$

as $e \to \infty$, while if $k = -\ell < 0$, then

$$\nu_2 = e - 1 + d_2(a-b-1) - d_2(\ell-1) - (e - 2 + d_2(2a-b-1) - d_2(\ell)) + d_2(2^{e-2}b - 1) \to \infty.$$

The odd-primary case follows similarly.

The second case of the theorem follows from the result for $c = 0$ just established and (1.2) by induction. For the third case, write $c = -k$ and $d = -\ell$ and argue by induction on k and ℓ, starting with the fact that the result is true if $k = 0$ or $l = 0$. Then, $\bmod p^e$,

$$\begin{aligned} S(p^e a - k - 1, p^e b - \ell - 1) &= S(p^e a - k, p^e b - \ell) - (p^e b - \ell) S(p^e a - k - 1, p^e b - \ell) \\ &\equiv S(p^e a, p^e b)(|s(\ell, k)| + \ell |s(\ell, k+1)|) \\ &= S(p^e a, p^e b)|s(\ell + 1, k + 1)|, \end{aligned}$$

implying the result. □

The proof of Theorem 1.4 will utilize the following two lemmas. We let $\lg_p(x) = [\log_p(x)]$.

Lemma 2.3. *If p is any prime and k and d are positive integers, then*

$$\nu_p\big(T_p((p-1)k + d - 1, d) - T_p(d-1, d)\big) \geq \nu_p(k) - \lg_p(d).$$

Proof. We have

$$\begin{aligned} &|T_p((p-1)k + d - 1, d) - T_p(d-1, d)| \\ &= \sum_{r=1}^{p-1} (-1)^r \tfrac{1}{d!} \sum_j (-1)^j \tbinom{d}{pj+r} (pj+r)^{d-1}((pj+r)^{(p-1)k} - 1) \\ &= \sum_{r=1}^{p-1} (-1)^r \sum_{i>0, t\geq 0} r^{(p-1)k+d-1-i-t} \tbinom{(p-1)k}{i} \tbinom{d-1}{t} \tfrac{1}{d!} \sum_j (-1)^j \tbinom{d}{pj+r} (pj)^{i+t}. \end{aligned}$$

Since $\binom{(p-1)k}{i} = \frac{(p-1)k}{i}\binom{(p-1)k-1}{i-1}$, we have $\nu_p\binom{(p-1)k}{i} \ge \nu_p(k) - \nu_p(i)$ for $i > 0$. Also

$$\nu_p\Big(\tfrac{1}{d!}\sum_j (-1)^j \tbinom{d}{pj+r}(pj)^{i+t}\Big) \ge \max(0, i+t-\nu_p(d!)),$$

with the first part following from [8, Theorem 1.1]. Thus it will suffice to show

$$\mathrm{lg}_p(d) - \nu_p(i) + \max(0, i+t-\nu_p(d!)) \ge 0.$$

This is clearly true if $\nu_p(i) \le \mathrm{lg}_p(d)$, while if $\nu_p(i) > \mathrm{lg}_p(d) = \ell$, then $\nu_p(d!) \le \nu_p((p^{\ell+1}-1)!) = \frac{p^{\ell+1}-1}{p-1} - \ell - 1$ and $i - \nu_p(i) \ge p^{\ell+1} - \ell - 1$, implying the lemma. $\square$

The following lemma is easily proved by induction on A.

Lemma 2.4. *If A and B are positive integers, then*

$$\sum_{i=0}^{A-1} \tbinom{i+B-1}{i} = \tbinom{A+B-1}{B}.$$

Now we can prove Theorem 1.4. We first prove it when $p = 2$, and then indicate the minor changes required when p is odd. Using Theorem 2.2 at the first step and Theorem 2.1 at the second, we have

$$\begin{aligned}
&S(2^e a + d - 1, 2^e b + d)\\
&\equiv \sum_{i=2^e b}^{2^e a - 1} S(i, 2^e b) S(2^e a + d - 1 - i, d) \bmod 2^e\\
&\equiv \sum_{j=2^{e-1}b}^{2^{e-1}a-1} \binom{j - 2^{e-2}b - 1}{j - 2^{e-1}b} S(2^e a + d - 1 - 2j, d) \bmod 2^{e-1}\\
&= \sum_{k=0}^{2^{e-1}(a-b)-1} \binom{k + 2^{e-2}b - 1}{k} S(2^e(a-b) + d - 1 - 2k, d)\\
&= \sum_{\ell=1}^{2^{e-1}(a-b)} \binom{2^{e-2}(2a-b) - 1 - \ell}{2^{e-2}b - 1} S(2\ell + d - 1, d)\\
&= \sum_{\ell=1}^{2^{e-1}(a-b)} \binom{2^{e-2}(2a-b) - 1 - \ell}{2^{e-2}b - 1} \Big(T_2(2\ell + d - 1, d) \pm \frac{1}{d!}\sum_j \binom{d}{2j}(2j)^{2\ell+d-1}\Big).
\end{aligned}$$

We have $\nu_2\binom{2^{e-2}(2a-b)-1-\ell}{2^{e-2}b-1} = f(a,b) + e - \nu_2(\ell)$, where $f(a,b) = \nu_2\binom{2a-b-1}{2a-2b} + \nu_2(a-b) - 1$. By [5, Theorem 1.5],

$$\nu_2\Big(\tfrac{1}{d!}\sum \tbinom{d}{2j}(2j)^{2\ell+d-1}\Big) \ge 2\ell + \tfrac{d}{2} - 1. \tag{2.1}$$

Thus, using Lemma 2.3 at the first step and Lemma 2.4 at the second, we obtain

$$\begin{aligned}
&S(2^e a+d-1, 2^e b+d)\\
&\equiv T_2(d-1,d)\sum_{k=0}^{2^{e-1}(a-b)-1}\binom{k+2^{e-2}b-1}{k} \bmod 2^{\min(e-1,e+f(a,b)-\lg(d))}\\
&= T_2(d-1,d)\binom{2^{e-1}(a-b)+2^{e-2}b-1}{2^{e-2}b}.
\end{aligned}$$

Letting $e\to\infty$ yields the claim of Theorem 1.4. In the congruence, we have also used that $\nu_2(T_2(d-1,d))\geq 0$. In fact, by (2.1) and $S(d-1,d)=0$, we have $\nu_2(T_2(d-1,d))\geq \frac{d}{2}-1$. See Table 2 for some explicit values of $T_2(d-1,d)$.

We now present the minor modifications required when p is odd and $a\equiv b \bmod (p-1)$. Let $a=b+(p-1)t$. Then

$$\begin{aligned}
&S(p^e a+d-1, p^e b+d)\\
&\equiv \sum_{j=0}^{p^e t-1} S(p^e b+(p-1)j, p^e b)S(p^e(a-b)-(p-1)j+d-1,d)\\
&\equiv \sum_{j=0}^{p^e t-1}\binom{p^{e-1}b+j-1}{j}S(p^e(p-1)t-(p-1)j+d-1,d)\\
&= \sum_{\ell=1}^{p^e t}\binom{p^e t+p^{e-1}b-\ell-1}{p^{e-1}b-1}S((p-1)\ell+d-1,d)\\
&\equiv T_p(d-1,d)\sum_{j=0}^{p^e t-1}\binom{p^{e-1}b+j-1}{j}\\
&= T_p(d-1,d)\binom{p^e t+p^{e-1}b-1}{p^{e-1}b}.
\end{aligned}$$

3. More Formulas and Numerical Values

In Theorem 1.3, we gave a simple formula for $S(p^\infty a+c, p^\infty b+d)$ when $d\leq 0$. For $d>0$, all values can be written explicitly using (1.2) and the initial values given in Theorem 1.4, provided $a\equiv b \bmod (p-1)$.

First assume $c\geq d-1$. For $i\geq 1$, define Stirling-like numbers $S_i(c,d)$ satisfying that for $d<i$ or $c\leq d-1$ the only nonzero value is $S_i(i-1,i)=1$ and satisfying the analogue of (1.2) when $c\geq d$. Note that $S_1(c,d)=S(c,d)$ if $(c,d)\notin\{(0,0),(0,1)\}$. The following result is easily obtained. Here we use that the binomial coefficient in Theorem 1.4 equals $\frac{p}{p-1}\frac{a-b}{b}S(p^\infty a, p^\infty b)$.

Proposition 3.1. *Assume $a\equiv b \bmod (p-1)$. For $d\geq 1$, $c\geq d-1$, we have*

$$S(p^\infty a+c, p^\infty b+d) = S(p^\infty a, p^\infty b)\big(S(c,d)+\sum_{i=1}^{d} S_i(c,d)T_p(i-1,i)\tfrac{p}{p-1}\tfrac{a-b}{b}\big).$$

The reader may obtain a better feeling for these numbers from the table of values of $S(p^\infty a+c, p^\infty b+d)/S(p^\infty a, p^\infty b)$ in Table 1, in which T_i denotes $T_p(i-1,i)\frac{p}{p-1}\frac{a-b}{b}$.

Table 1. $S(p^\infty a + c, p^\infty b + d)/S(p^\infty a, p^\infty b)$ when $a \equiv b \bmod (p-1)$

		d				
		1	2	3	4	5
	0	T_1				
	1	$1+T_1$	T_2			
c	2	$1+T_1$	$1+T_1+2T_2$	T_3		
	3	$1+T_1$	$3+3T_1+4T_2$	$1+T_1+2T_2$ $+3T_3$	T_4	
	4	$1+T_1$	$7+7T_1+8T_2$	$6+6T_1$ $+10T_2+9T_3$	$1+T_1+2T_2$ $+3T_3+4T_4$	T_5
	5	$1+T_1$	$15+15T_1$ $+16T_2$	$25+25T_1$ $+38T_2+27T_3$	$10+10T_1+18T_2$ $+21T_3+16T_4$	$1+T_1+2T_2$ $+3T_3+4T_4$ $+5T_5$

The first few values of $T_2(d-1,d)$ and $T_3(d-1,d)$ are given in Table 2.

Table 2. Some values of $T_2(d-1,d)$ and $T_3(d-1,d)$

d	1	2	3	4	5	6	7	8
$T_2(d-1,d)$	1	-1	2	$-\frac{14}{3}$	12	$-\frac{164}{5}$	$\frac{4208}{5}$	$-\frac{86608}{315}$
$T_3(d-1,d)$	1	0	$-\frac{3}{2}$	$\frac{9}{2}$	$-\frac{27}{4}$	$-\frac{81}{20}$	$\frac{4779}{80}$	$-\frac{15309}{80}$

For $c < d-1$, we use (1.2) to work backwards from $S(p^\infty a + d - 1, p^\infty b + d)$, obtaining the following proposition.

Proposition 3.2. *Suppose $a \equiv b \bmod (p-1)$. For $k \geq 1$, $d \geq 0$, let $Y(k,d) = S(p^\infty a + d - k, p^\infty b + d)$. Then $Y(1,d)$ is as in Theorem 1.4 for $d \geq 1$, $Y(k,0) = 0$ for $k \geq 1$, and, for $k \geq 2$, $d \geq 1$,*

$$Y(k,d) = \big(Y(k-1,d) - Y(k-1,d-1)\big)/d.$$

We illustrate these values in Table 3, where again T_i denotes $T_p(i-1,i)\frac{p}{p-1}\frac{a-b}{b}$.

Note that since $S(d-1,d) = 0$ and $T_p(n,k) - S(n,k)$ is a sum like that in (1.1) taken over $i \equiv 0 \bmod p$, we deduce that $T_p(d-1,d) = 0$ if $1 < d < p$, which simplifies these results slightly.

TABLE 3. $S(p^\infty a + c, p^\infty b + d)/S(p^\infty a, p^\infty b)$ when $a \equiv b \bmod (p-1)$

		d			
		1	2	3	4
	-2	T_1	$\frac{1}{8}T_2 - \frac{7}{8}T_1$	$\frac{1}{81}T_3 - \frac{65}{648}T_2 + \frac{85}{216}T_1$	$\frac{1}{1024}T_4 - \frac{781}{82944}T_3 + \frac{865}{20736}T_2 - \frac{415}{3456}T_1$
	-1	T_1	$\frac{1}{4}T_2 - \frac{3}{4}T_1$	$\frac{1}{27}T_3 - \frac{19}{108}T_2 + \frac{11}{36}T_1$	$\frac{1}{256}T_4 - \frac{175}{6912}T_3 + \frac{115}{1728}T_2 - \frac{25}{288}T_1$
c	0	T_1	$\frac{1}{2}T_2 - \frac{1}{2}T_1$	$\frac{1}{9}T_3 - \frac{5}{18}T_2 + \frac{1}{6}T_1$	$\frac{1}{64}T_4 - \frac{37}{576}T_3 + \frac{13}{144}T_2 - \frac{1}{24}T_1$
	1		T_2	$\frac{1}{3}T_3 - \frac{1}{3}T_2$	$\frac{1}{16}T_4 - \frac{7}{48}T_3 + \frac{1}{12}T_2$
	2			T_3	$\frac{1}{4}T_4 - \frac{1}{4}T_3$

4. THE CASE $a \not\equiv b \bmod (p-1)$

In this section, we complete the proof of Theorem 1.1 when $a \not\equiv b \bmod (p-1)$ by proving the following case.

Theorem 4.1. *Suppose $0 < b \le a$ and $d \ge 1$. Then the p-adic limit of $S(p^{e+1}a - (a-b), p^{e+1}b + d)$ exists as $e \to \infty$.*

Then $\lim_e S(p^{e+1}a + c, p^{e+1}b + d)$ exists for all integers c by induction using (1.2).

Let $R_p(e) = (p^{e+1} - 1)/(p-1)$. The proof of Theorem 4.1 begins with, $\bmod p^e$,

$$\begin{aligned}
&S(p^{e+1}a - (a-b), p^{e+1}b + d)\\
&\equiv \sum_{j=0}^{R_p(e)(a-b)} S(p^{e+1}b + (p-1)j, p^{e+1}b) S((p^{e+1}-1)(a-b) - (p-1)j, d)\\
&\equiv \sum_{j=0}^{R_p(e)(a-b)} \binom{p^e b + j - 1}{j} \frac{(-1)^d}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} i^{(p^{e+1}-1)(a-b)-(p-1)j}\\
&= \sum_{i=0}^{d} (-1)^{i+d} \frac{1}{d!} \binom{d}{i} \sum_{j=0}^{R_p(e)(a-b)} \binom{p^e b + j - 1}{j} i^{(p^{e+1}-1)(a-b)-(p-1)j}.
\end{aligned}$$

We show that for each i, the limit as $e \to \infty$ of

$$\sum_{j=0}^{R_p(e)(a-b)} \binom{p^e b + j - 1}{j} i^{(p^{e+1}-1)(a-b)-(p-1)j} \tag{4.1}$$

exists in $\mathbb{Z}_p$. This will complete the proof of the theorem.

If $i \not\equiv 0 \bmod p$, write $i^{p-1} = Ap + 1$, using Fermat's Little Theorem. Then (4.1) becomes

$$\begin{aligned}
&\sum_{\ell=0}^{R_p(e)(a-b)} (Ap)^\ell \sum_{j=0}^{R_p(e)(a-b)} \binom{p^e b + j - 1}{j} \binom{R_p(e)(a-b) - j}{\ell}\\
&= \sum_{\ell=0}^{R_p(e)(a-b)} (Ap)^\ell \binom{p^e b + R_p(e)(a-b)}{p^e b + \ell}
\end{aligned}$$

by [7, p.9(3c)]. Lemma 4.3 says that for each ℓ, there exists a p-adic integer

$$z_\ell := \lim_{e\to\infty} \binom{p^e b + R_p(e)(a-b)}{p^e b + \ell}.$$

Then $\sum_{\ell=0}^{\infty} (Ap)^\ell z_\ell$ is a p-adic integer, which is the limit of (4.1) as $e \to \infty$.

If $i = 0$, since $0^0 = 1$ in (4.1) and the equations preceding it, (4.1) becomes

$$\binom{p^e b + R_p(e)(a-b) - 1}{p^e b - 1} = \frac{p^e b}{p^e b + R_p(e)(a-b)} \binom{p^e b + R_p(e)(a-b)}{p^e b}.$$

Since by the proof of Lemma 4.3 $\nu_p\binom{p^e b + R_p(e)(a-b)}{p^e b}$ is eventually constant, $\binom{p^e b + R_p(e)(a-b)-1}{p^e b-1} \to 0$ in $\mathbb{Z}_p$, due to the $p^e b$ factor.

We complete the proof of Theorem 4.1 in the following lemma, which shows that the p-adic limit of (4.1) is 0 when $i \equiv 0 \bmod p$ and $i > 0$.

Lemma 4.2. *If $0 \le j \le R_p(e)(a-b)$, then*

$$\nu_p \binom{p^e b + j - 1}{j} + (p^{e+1} - 1)(a-b) - (p-1)j \ge e - \log_p(a - b + p)$$

for e sufficiently large.

Proof. Let $\ell = R_p(e)(a-b) - j$ and $a - b = (p-1)t + \Delta$, $1 \le \Delta \le p-1$. The p-exponent of the binomial coefficient becomes

$$d_p(b-1) + e + d_p((p^{e+1}-1)t + R_p(e)\Delta - \ell) - d_p((p^{e+1}-1)t + R_p(e)\Delta + p^e b - \ell - 1). \quad (4.2)$$

Choose s minimal so that $\frac{\Delta}{p-1}(p^s - 1) - \ell - 1 - t \ge 0$. Then, if $e > s$, the p-ary expansion of $(p^{e+1}-1)t + R_p(e)\Delta - \ell$ splits as

$$p^e(pt + \Delta) + p^s \frac{p^{e-s} - 1}{p-1}\Delta + \frac{p^s - 1}{p-1}\Delta - \ell - t,$$

and there is a similar splitting for the expression at the end of (4.2). We obtain that (4.2) equals

$$e + \nu_p(b) + \nu_p\tbinom{pt+b+\Delta}{b} - \nu_p\left(\tfrac{\Delta}{p-1}(p^s-1) - \ell - t\right).$$

The expression in the lemma equals this plus $(p-1)\ell$. Since s was minimal, we have $\frac{\Delta}{p-1}(p^s - 1) - \ell - t \le (p-1)(\ell+t) + p + \Delta$, and hence, $\nu_p(\frac{\Delta}{p-1}(p^s-1) - \ell - t) \le \log_p((p-1)(\ell+t)+p+\Delta)$. The smallest value of $(p-1)\ell - \log_p((p-1)(\ell+t)+p+\Delta)$ occurs when $\ell = 0$. We obtain that the expression in the lemma is $\ge e - \log_p(a-b+p)$. $\square$

The following lemma was referred to above.

Lemma 4.3. *If α and b are positive integers and $\ell \ge 0$, then*

$$\lim_{e\to\infty} \binom{p^e b + R_p(e)\alpha}{p^e b + \ell}$$

exists in $\mathbb{Z}_p$.

This is another p-adic binomial coefficient, slightly different than those of [4], which we would call $\binom{p^\infty b+R_p(\infty)\alpha}{P^\infty b+\ell}$. The proof of the lemma breaks into two parts: showing that the p-exponents are eventually constant, and showing that the unit parts approach a limit.

The proof that the p-exponent is eventually constant is very similar to the proof of Lemma 4.2. Let $\alpha=(p-1)t+\Delta$ with $1\le\Delta\le p-1$, and choose s minimal such that $\frac{\Delta}{p-1}(p^s-1)-t-\ell\ge 0$. Then the p-ary expansions split again into three parts and we obtain that for $e>s$, the desired p-exponent equals $\nu_p\binom{pt+b+\Delta}{b}+\nu_p\binom{\Delta(p^s-1)/(p-1)-t}{\ell}$, independent of e.

We complete the proof of Lemma 4.3 by showing that, if $\ell<\min(R_p(e-1)\alpha,p^eb)$ and $p^e>\alpha$, then

$$\mathrm{U}_p\binom{p^{e-1}b+R_p(e-1)\alpha}{p^{e-1}b+\ell}\equiv\mathrm{U}_p\binom{p^eb+R_p(e)\alpha}{p^eb+\ell}\bmod p^{e+f(\alpha,b,\ell)-1},\tag{4.3}$$

where $f(\alpha,b,\ell)=\min(\nu_p(b)-\mathrm{lg}_p(\alpha),\nu_p(\alpha)-\mathrm{lg}_p(\ell),\nu_p(b)-\mathrm{lg}_p(\ell),1)$. We write the second binomial coefficient in (4.3) as

$$(-1)^{eb}\frac{(p^eb+R_p(e)\alpha)!}{(R_p(e)\alpha)!}\cdot\frac{(R_p(e)\alpha)!}{(R_p(e)\alpha-\ell)!}\cdot\frac{(p^eb)!}{(p^eb+\ell)!}\cdot\frac{(-1)^{eb}}{(p^eb)!}.\tag{4.4}$$

We show that the unit parts of these four factors are congruent to their $(e-1)$-analogue mod $p^{e+\nu_p(b)-\mathrm{lg}(\alpha)-1}$, $p^{e+\nu_p(\alpha)-\mathrm{lg}_p(\ell)-1}$, $p^{e+\nu_p(b)-\mathrm{lg}_p(\ell)-1}$, and p^e, respectively, which will imply the result. For the fourth factor, this was shown in [4]. For the second and third, the claim is clear, since each of the ℓ unit factors being multiplied will be congruent to their $(e-1)$-analogue modulo the specified amount.

To prove the first, we will prove

$$\mathrm{U}_p\left(\frac{(R_p(e)\alpha+1)\cdots(R_p(e)\alpha+p^eb)}{(R_p(e-1)\alpha+1)\cdots(R_p(e-1)\alpha+p^{e-1}b)}\right)\equiv(-1)^b\bmod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}.\tag{4.5}$$

Since $\mathrm{U}_p(j)=\mathrm{U}_p(pj)$, we may cancel most multiples of p in the numerator with factors in the denominator. Using that $p\cdot R_p(e-1)=R_p(e)-1$, we obtain that the LHS of (4.5) equals $P\,\mathrm{U}_p(A)/\,\mathrm{U}_p(B)$, where P is the product of the units in the numerator, A is the product of all $j\equiv 0\bmod p$ which satisfy

$$(R_p(e)-1)\alpha+p^eb<j\le R_p(e)\alpha+p^eb,$$

and B is the product of all integers k such that

$$R_p(e-1)\alpha+1\le k\le R_p(e-1)\alpha+\left[\tfrac{\alpha}{p}\right].\tag{4.6}$$

Since the mod p^e values of the p-adic units in any interval of p^e consecutive integers are just a permutation of the set of positive p-adic units less than p^e, and by [6, Lemma 1] the product of these is $-1\bmod p^e$, we obtain $P\equiv(-1)^b\bmod p^e$. Thus (4.5) reduces to showing $\mathrm{U}_p(A)/\,\mathrm{U}_p(B)\equiv 1\bmod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}$.

We have

$$\frac{\mathrm{U}_p(A)}{\mathrm{U}_p(B)}=\prod\frac{\mathrm{U}_p(k+p^{e-1}b)}{\mathrm{U}_p(k)},$$

taken over all k satisfying (4.6). We show that if k satisfies (4.6), then

$$\nu_p(k)\le\mathrm{lg}_p(\alpha).\tag{4.7}$$

Then $\mathrm{U}_p(k)\equiv\mathrm{U}_p(k+p^{e-1}b)\bmod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}$, establishing the result.

We prove (4.7) by showing that it is impossible to have $1 \le \alpha < p^t$, $1 \le i \le [\frac{\alpha}{p}]$, $t < e$, and

$$R_p(e-1)\alpha + i \equiv 0 \bmod p^t. \tag{4.8}$$

From (4.8) we deduce $\alpha \equiv i(p-1) \bmod p^t$. But $i(p-1) < \alpha$, so the only way to satisfy (4.8) would be with $\alpha = p^t$ and $i = 0$, but $\alpha < p^t$.

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MSC2010: 11B73, 11A07.

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