ON CERTAIN SERIES EXPANSIONS OF THE SINE FUNCTION: CATALAN NUMBERS AND CONVERGENCE

PETER J. LARCOMBE, SAM T. O'NEILL, AND ERIC J. FENNESSEY

Abstract. The appearance of Catalan numbers in certain infinite series expansions of the sine function was first reported well over a decade ago. A combination of computation and analysis is employed as we return to this topic and examine the outstanding issue of convergence for this suite of results and also for the general case expansion.

1. Introduction and Background

The well-known Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$, with $(n+1)$th term

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

has its origins in 18th century China through discovery by scholar Antu Ming who found what we know as Catalan numbers occurring in some expansions of the sine function. In 2000 Larcombe gave a historical backdrop to these series and established formally the initial results [2, Results I, II, pp. 41, 42]

$$\sin(2\alpha) = 2\left\{\sin(\alpha) - \sum_{n=1}^{\infty}\left[\frac{c_{n-1}}{2^{2n-1}}\right]\sin^{2n+1}(\alpha)\right\} \tag{1.2}$$

and

$$\sin(4\alpha) = 2\left\{2\sin(\alpha) - 5\sin^3(\alpha) + \sum_{n=1}^{\infty}\left[\frac{8c_{n-1} - c_n}{4^n}\right]\sin^{2n+3}(\alpha)\right\}, \tag{1.3}$$

identifying a recursive procedure to allow extension of these and giving results accordingly for $\sin(6\alpha), \sin(8\alpha)$ and $\sin(10\alpha)$. Assuming a non-zero frequency n, say, consideration of the classic differential equation $0 = y''(z) + n^2 y(z)$ governing simple harmonic motion for $y(z)$ led a short time later to the general series [5, Eq. (18), p. 68]

$$\sin(nz) = n\sin(z)\,{}_2F_1\left(\begin{matrix}\frac{1}{2} - \frac{n}{2}, \frac{1}{2} + \frac{n}{2} \\ \frac{3}{2}\end{matrix}\,\middle|\,\sin^2(z)\right), \tag{1.4}$$

with the special case $n = 2$ (and $z = \alpha$) shown to deliver (1.2) as a point of interest; the wider role of hypergeometric function theory in these types of expansions was also discussed.

Given integer $p \geq 1$, a general form of the expansion for $\sin(2p\alpha)$, in odd powers of $\sin(\alpha)$, was proposed [2] in view of the particular results developed therein. This was picked up and analyzed by Xinrong [7] whose starting point was a standardized form

$$\sin(2p\alpha) = 2\left\{\sum_{n=1}^{p}\alpha_n^{(p)}\sin^{2n-1}(\alpha) + \sum_{n=1}^{\infty}\frac{h_p(c_{n-1}, \ldots, c_{n+p-2})}{2^{2(n+p)-3}}\sin^{2(n+p)-1}(\alpha)\right\} \tag{1.5}$$

(of which (1.2) and (1.3) are special cases with $\alpha_1^{(1)} = 1$, $h_1(c_{n-1}) = -c_{n-1}$ and $\alpha_1^{(2)} = 2$, $\alpha_2^{(2)} = -5$, $h_2(c_{n-1}, c_n) = 2(8c_{n-1} - c_n)$). He confirmed that—beyond an initial p stand-alone terms with individual numerical coefficients—the functional coefficient $h_p(c_{n-1}, \ldots, c_{n+p-2})$ of each remaining term in the expansion indeed had the key interesting feature that it comprised a linear combination of the p Catalan elements $c_{n-1}, \ldots, c_{n+p-2}$ (Larcombe [3, Eq. (13), p. 212] lists $h_3(c_{n-1}, c_n, c_{n+1}) = -(256c_{n-1} - 64c_n + 3c_{n+1})$, $h_4(c_{n-1}, c_n, c_{n+1}, c_{n+2}) = 4(1024c_{n-1} - 384c_n + 40c_{n+1} - c_{n+2})$ and $h_5(c_{n-1}, c_n, c_{n+1}, c_{n+2}, c_{n+3}) = -(65536c_{n-1} - 32768c_n + 5376c_{n+1} - 320c_{n+2} + 5c_{n+3})$). Writing

$$h_p(c_{n-1}, \ldots, c_{n+p-2}) = \beta_0^{(p)} c_{n-1} + \beta_1^{(p)} c_n + \cdots + \beta_{p-1}^{(p)} c_{n+p-2}, \tag{1.6}$$

he applied umbral calculus to yield its general coefficient as

$$\beta_n^{(p)} = [x^n]\{H_p(x)\}, \qquad n = 0, \ldots, p-1, \tag{1.7}$$

where (note the sign factor omitted in [7, Theorem 1.2, p. 158])

$$H_p(x) = (-1)^p \frac{(2 + \sqrt{4-x})^{2p} - (2 - \sqrt{4-x})^{2p}}{8\sqrt{4-x}} \tag{1.8}$$

is, for $p \geq 1$, a degree $p-1$ polynomial in x which acts as an (ordinary) generating function for the finite p-sequence $\{\beta_0^{(p)}, \ldots, \beta_{p-1}^{(p)}\}$ of coefficients. A preprint of [7], sent to the author P. J. L. in early 2001 as a private communication from Xinrong, allowed this observation to be carried forward with an intermediate and final closed form for $\beta_n^{(p)}$ formulated thus [3, Lemmas 1, 2, pp. 213, 216]:

$$\beta_n^{(p)} = (-1)^{n+p} 2^{2(p-n)-3} \sum_{i=n}^{p-1} \binom{i}{n} \binom{2p}{2i+1} = (-1)^{n+p} 16^{p-(n+1)} \binom{2p-(n+1)}{n}. \tag{1.9}$$

With $\beta_n^{(p)}$ determined, the function (linear sum) $h_p(c_{n-1}, \ldots, c_{n+p-2}) = \sum_{i=0}^{p-1} \beta_i^{(p)} c_{n+i-1}$ of (1.6) can be converted to hypergeometric form and duly evaluated [3, Theorem, p. 218] as

$$h_p(c_{n-1}) = (-1)^p p(n+p) \frac{[2(n+2p-1)]!n!}{(n+2p-1)![2(n+p)]!} c_{n-1}. \tag{1.10}$$

The structure of (1.10) reflects the potential repeated use (in a 'cascading' manner) of the simple recurrence $(n+2)c_{n+1} = 2(2n+1)c_n$, relating neighboring Catalan numbers, to reduce the dependency of h_p on p Catalan numbers to just one, namely, c_{n-1}. While it is impossible to execute this procedure and arrive at (1.10) other than for a few low values of p, hypergeometric conversion and evaluation of h_p (1.6) yields it entirely naturally.

The appearance of the Catalan numbers in the general expansion (1.5) is, to say the least, somewhat surprising, and is but one more instance of the ubiquity of the Catalan sequence. Remarking for completeness that Chinese historian J. Luo was the first to announce Ming's awareness of Catalan numbers in 1988 [6] (see also [1]), and that Remark 6 of [4] offers some little-known biographical information on Ming himself, this concludes the historical and technical background to our presentation.

2. Results

The convergence interval $|z| < \pi/2$ for all even n instances of (1.4) emerges naturally from the methodology adopted in [5], and this interval was shown to apply (w.r.t. α) in (1.2), (1.3) and those other expansions given originally in [2]. Numeric confirmation, and a more detailed treatment of the issue of convergence within this class of series, together form the remit of the paper.

2.1. **Convergence over** $(-\frac{\pi}{2}, \frac{\pi}{2})$**.** Write the r.h.s. series of (1.5) as $R_p(\alpha)$. Noting that (1.5) holds trivially at $\alpha = 0$ ($R_p(0) = 0$), the parity of the sine function (being an odd one) guarantees that convergence of $R_p(\alpha)$ over the α interval $(0, \frac{\pi}{2})$ is matched exactly over $(-\frac{\pi}{2}, 0)$, and so convergence occurs over the full open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

2.2. **Endpoints Convergence.** Here we examine convergence at the interval endpoints $\alpha = \pm\frac{\pi}{2}$, where (1.5) reads

$$0 = \sum_{n=1}^{p} \alpha_n^{(p)} + \sum_{n=1}^{\infty} \frac{h_p(c_{n-1}, \ldots, c_{n+p-2})}{2^{2(n+p)-3}} = E(p), \tag{2.1}$$

say, in both sign cases. It is this equation we now wish to show holds in order to establish convergence, which of course requires the correct convergence of the infinite sum within $E(p)$; we write $a_p(n) = h_p(c_{n-1}, \ldots, c_{n+p-2})/2^{2(n+p)-3}$ $(n \geq 1)$, and thus consider $\sum_{n=1}^{\infty} a_p(n)$.

We adopt the form (1.10) of h_p for convenience, denoting $h_p(c_{n-1})$ by $h_p(n)$. With $a_p(n) = h_p(n)/2^{2(n+p)-3}$, this gives

$$\frac{a_p(n+1)}{a_p(n)} = \frac{1}{4}\frac{h_p(n+1)}{h_p(n)} = \frac{1}{2}\frac{(2n-1)(2n+4p-1)}{(n+p)(2n+2p+1)} \tag{2.2}$$

after some algebra, hence $\lim_{n\to\infty}\{|a_p(n+1)/a_p(n)|\} = \lim_{n\to\infty}\{a_p(n+1)/a_p(n)\} = 1$ and the Ratio Test is inconclusive in terms of convergence.[1] Similarly, the Limit (or nth term) Test renders convergence unresolved since it is known [3, Eq. (37), p. 219] that

$$h_p(n) \sim \frac{(-1)^p 4^{n+p-2} p}{\sqrt{\pi}} \frac{1}{n\sqrt{n}} \tag{2.3}$$

for large $n \gg p \geq 1$, so that $a_p(n) \sim (-1)^p p/2n\sqrt{\pi n} = O(1/n\sqrt{n})$ and $\lim_{n\to\infty}\{a_p(n)\} = 0$.

A different approach, however, proves more successful. Define an infinite sequence of terms $\{b_1(n), b_2(n), b_3(n), \ldots\}$, where, for $p \geq 1$, $b_p(n) = (-1)^p p/2n\sqrt{\pi n}$ (that is, the large n form of $a_p(n)$). Clearly, $\lim_{n\to\infty}\{|a_p(n)/b_p(n)|\} = \lim_{n\to\infty}\{a_p(n)/b_p(n)\} = 1$ so that the series $\sum_{n=1}^{\infty} a_p(n)$ and $\sum_{n=1}^{\infty} b_p(n)$ either both converge or diverge (Limit Comparison Test); in other words, $\sum_{n=1}^{\infty} a_p(n)$ converges if and only if $\sum_{n=1}^{\infty} 1/n\sqrt{n}$ converges. Noting that $n^{-3/2} > 0$ for $n \geq 1$, put $g(x) = x^{-3/2}$. Evidently, $g(x)$ is a decreasing and continuous function on the interval $[1, \infty)$, and since $\int_1^{\infty} g(x)\, dx$ has a finite value of 2 then $\sum_{n=1}^{\infty} 1/n\sqrt{n}$ converges by virtue of the Integral Test; hence, our sum $\sum_{n=1}^{\infty} a_p(n)$ also converges, and we conclude as follows. We rewrite $E(p)$ (2.1) in the form

$$E(p) = \sum_{n=1}^{p} \alpha_n^{(p)} + \sum_{n=1}^{\infty} a_p(n) = \sum_{n=1}^{p} e_n^{(p)} + \sum_{n=p+1}^{\infty} e_n^{(p)} = \sum_{n=1}^{\infty} e_n^{(p)}, \tag{2.4}$$

as a single sum series (where $e_n^{(p)} = \alpha_n^{(p)}$, $n = 1, \ldots, p$). Now, since $\sum_{n=1}^{\infty} a_p(n)$ converges so does $E(p)$, and it remains to show that $E(p)$ converges to zero which we are able to do without difficulty by appealing to Abel's Theorem (stated for completeness).

[1] The finite sum $h_p(c_{n-1}, \ldots, c_{n+p-2})$, as seen empirically from the explicit known cases for $p = 1, \ldots, 5$, delivers terms that are all positive or all negative depending on the parity of p; the appropriate sign is evident from the contracted univariate form $h_p(c_{n-1}) = h_p(n)$ (1.10). Thus, for fixed $p \geq 1$ then $\sum_{n=1}^{\infty} a_p(n)$ is a single sign series with term ratio $|a_p(n+1)/a_p(n)| = a_p(n+1)/a_p(n) > 0$ (positivity is also confirmed by (2.2)).

Theorem 2.1. (Abel's Theorem) *Let real $r > 0$ and suppose $\sum_{n=0}^{\infty} d_n r^n$ converges. Define, on the open interval $(-r, r)$ for x, a function $f(x) = \sum_{n=0}^{\infty} d_n x^n$. Then*

$$\lim_{x \to r} \{f(x)\} = \sum_{n=0}^{\infty} d_n r^n.$$

If we define a function $f(x) = \sum_{n=1}^{\infty} e_n^{(p)} x^{2n-1}$ (where $x = \sin(\alpha)$) on the open interval $(-1, 1)$ for x, then on this interval we must have $f(x) = \sin(2p\sin^{-1}(x))$ by construction. Applying Abel's Theorem duly gives $E(p) = \lim_{x\to 1}\{f(x)\} = \lim_{x \to 1}\{\sin(2p\sin^{-1}(x))\} = 0$, as required.

2.3. **Convergence over a Cycle.** The series expansion $R_p(\alpha)$ of (1.5) has been shown to hold at $\alpha = \pm\frac{\pi}{2}$, and the convergence interval for it can now be stated as $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Based on this, we are able to make observations about convergence over a complete cycles interval $[0, 2\pi]$ during which, having period π/p, the function $\sin(2p\alpha)$ completes $2p$ cycles with $p/2$ cycles over each of the four sub-intervals $[0, \frac{\pi}{2}], [\frac{\pi}{2}, \pi], [\pi, \frac{3\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$. Computations indicate that $R_p(\alpha)$ converges correctly to $\sin(2p\alpha)$ over $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$, while over $(\frac{\pi}{2}, \pi)$ and $(\pi, \frac{3\pi}{2})$ convergence is to $-\sin(2p\alpha)$ which is, perhaps, counter-intuitive; Figure 1 illustrates this for values of $p = 1, 3$.

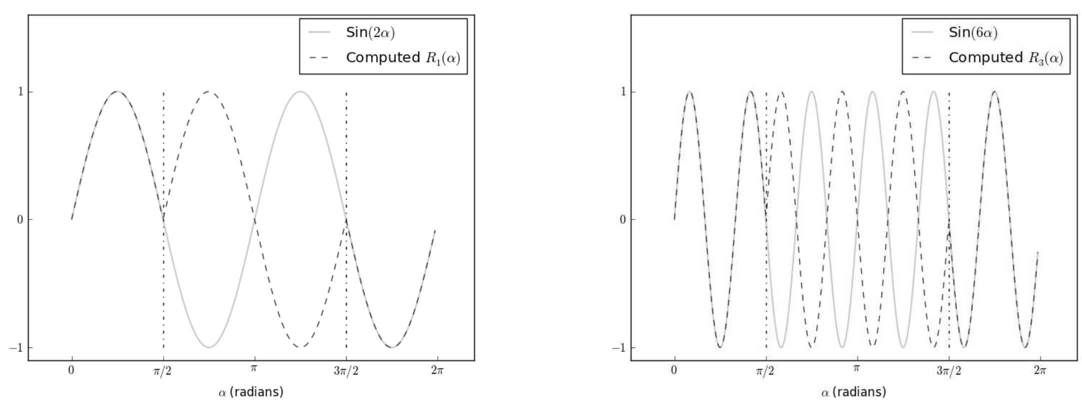

FIGURE 1. Plots of $R_1(\alpha)$ and $R_3(\alpha)$ versus α over $(0, 2\pi)$.

The observations are readily explained, however, as an immediate consequence of the parity of the sine function, for defining a function $f_p(\alpha) = \sum_{i=0}^{\infty} \gamma_i^{(p)} \sin^{2i+1}(\alpha)$ it is elementary to show that

$$f_p(\alpha + k\pi) \quad = \quad \begin{cases} f_p(\alpha) & k \text{ (even)} = 0, \pm 2, \pm 4, \ldots \\ -f_p(\alpha) & k \text{ (odd)} = \pm 1, \pm 3, \pm 5, \ldots . \end{cases} \tag{2.5}$$

Since it is known that there exists constants $\gamma_0^{(p)}, \gamma_1^{(p)}, \gamma_2^{(p)}, \ldots,$ such that $f_p(\alpha) = R_p(\alpha)$, we see precisely why $R_p(\alpha)$ converges correctly over $(0, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$ and, excepting $\alpha = \pi$ and other zeros of $\sin(2p\alpha)$ (all occurring at the $2p-1$ multiples $p+1, p+2, \ldots, 2p, \ldots, 3p-2, 3p-1$ of $\pi/2p$), incorrectly over $(\frac{\pi}{2}, \frac{3\pi}{2})$.

2.4. **Convergence Rate.** Note that, in view of (2.5), the original left-hand interval endpoint of consideration $\alpha = -\frac{\pi}{2}$ of Section 2.1 now translates to $\alpha = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$ which, along with $\alpha = \frac{\pi}{2}$, are those particular points where convergence of $R_p(\alpha)$ is more difficult to achieve computationally; Figure 2 shows an iteration count for the cases $p = 1, 2$ where this phenomenon is visibly evident. Away from the points $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}$ a high level of accuracy can be reached, 'convergence' of a series $R_p(\alpha)$ declared when successive values arising from the addition of a series term differ in magnitude by less than a tolerance of 10^{-10} (see also Section 4).

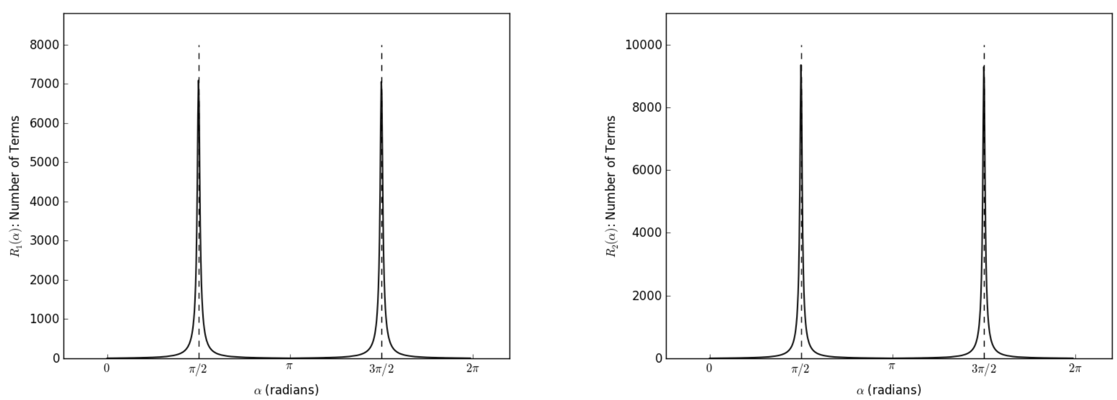

FIGURE 2. Plots of Computational Convergence Effort for $R_1(\alpha)$ and $R_2(\alpha)$ versus α over $[0, 2\pi]$.

3. AN EXPANSION BY EULER

In [4] a global expansion (said to be due to Euler)

$$\begin{aligned}\sin(m\alpha) &= m\sin(\alpha) - \left[\frac{m(m^2-1^2)}{3!}\right]\sin^3(\alpha)\\ &\quad + \left[\frac{m(m^2-1^2)(m^2-3^2)}{5!}\right]\sin^5(\alpha)\\ &\quad - \left[\frac{m(m^2-1^2)(m^2-3^2)(m^2-5^2)}{7!}\right]\sin^7(\alpha) + \cdots\\ &= \sum_{n=0}^{\infty} S_n^{(m)}\sin^{2n+1}(\alpha), \end{aligned} \tag{3.1}$$

was put forward (avoiding the split sum format of (1.5)), based on which the general (functional) coefficient is, for m (even) ≥ 2, $n \geq 0$, expressible in terms of Gamma functions [4, Lemma 2, p. 9] and in a form analogous to $h_p(c_{n-1})$ (1.10) [4, Theorem 2, p. 11]; the Section 2.2 convergence of $\sin(m\alpha)$ at $\alpha = \pm\pi/2$ could, of course, equally have been established using properties of $S_n^{(m)}$.

It is not known to the authors how Euler formulated his expansion (his methods of the day were sometimes mathematically extemporaneous), and the striking feature of (3.1)—which shows immediately why, for m (odd) ≥ 1, expansions are finite—is that it gives no hint of the potential role of Catalan numbers in r.h.s. coefficients for m even. For completeness, and as a point of interest to the general readership, we provide a pleasing derivation outline of Euler's expansion in an Appendix.

We finish with some remarks on computational aspects of the work presented, the large scale calculations incurred having at times caused convergence issues which we feel are worth a mention.

4. Computations

The graphs above were produced using the (interpreted) language Python (v. 2.7) in conjunction with the graphing library Matplotlib (v. 1.2.1) using a 64-bit (8 GB) operating system driven by an Intel Core i7 CPU; convergence was set as described above. While this set up was sufficient for the most part, it proved inadequate for those calculations performed very close to the points $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}$, since without relaxing the tolerance upwards an unrealistic amount of series terms are required in order to obtain results as these threshold points are approached. Under such circumstances individual terms also become almost impossible to evaluate due to the magnitude of the Catalan numbers, causing numeric intractability and unacceptable convergence rates (as an example, in the $\sin(2\alpha)$ expansion with a value of $\alpha = 1.57$ rads ($\pi/2 = 1.5708$) then using, for instance, $s = O(10^5)$ terms Catalan numbers c_s appear in calculations and series values of $\sin(2\alpha)$ differ from the true value by as much as 10^{-4} still).

Near to these points of interest it is clear that a so-called unbounded data type is needed to correctly hold the approximation in memory and achieve convergence to a high degree of accuracy. Python—being an interpreted language—struggles to provide the computational speed/memory required around these points. An alternative approach would be to use a compiled language, such as C++ or Java, which would certainly be much faster and allow for better approximations to be made here.

5. Summary

This paper completes an investigation into series expansions first motivated by the appearance of Catalan numbers in a historical context. Drawing on previous work, convergence of the expansion for $\sin(2p\alpha)$ has been examined analytically in the general case and computationally for low values of p. A corresponding expansion associated with Euler has also been discussed and formulated.

Appendix

Here we give a first principles method to obtain the expansion of Euler (3.1). Write $f(\alpha) = \sin(m\alpha)$ as $f(\alpha) = \sin(m\sin^{-1}[\sin(\alpha)])$. Setting $x = x(\alpha) = \sin(\alpha)$ then $f(\alpha) = f(x(\alpha)) = \sin(m\sin^{-1}[x(\alpha)])$. We seek a series form of $f(x) = \sin(m\sin^{-1}(x))$ in odd powers of x.

Differentiating w.r.t. x we find that $f'(x) = m(1-x^2)^{-1/2}\cos(m\sin^{-1}(x))$, and in turn

$$\begin{aligned} f''(x) &= mx(1-x^2)^{-3/2}\cos(m\sin^{-1}(x)) - m^2(1-x^2)^{-1}\sin(m\sin^{-1}(x)) \\ &= x(1-x^2)^{-1}f'(x) - m^2(1-x^2)^{-1}f(x). \end{aligned} \tag{A.1}$$

In other words,

$$f''(x) - P(x)f'(x) + Q(x)f(x) = 0, \tag{A.2}$$

where $P(x) = x/(1-x^2)$, $Q(x) = m^2/(1-x^2)$. Since the denominators of $P(x), Q(x)$ are non-zero at $x = 0$ these functional coefficients are analytic there. Thus, $x = 0$ is an ordinary point of the differential equation

$$(1-x^2)f''(x) - xf'(x) + m^2 f(x) = 0, \tag{A.3}$$

about which we can seek, with justification, a power series solution $f(x) = \sum_{n=0}^{\infty} u_n x^n$.

Omitting the details (which we leave as a straightforward reader exercise), this series form of solution, when substituted into (A.3), eventually yields a single recurrence relation

$$u_{n+2} = -\frac{(m^2-n^2)}{(n+1)(n+2)}u_n, \qquad n \geq 0. \tag{A.4}$$

With $u_0 = f(0) = \sin(m\sin^{-1}(0)) = 0$, then $u_0 = u_2 = u_4 = u_6 = \cdots = 0$, as required. Noting that $u_1 = f'(0) = m(1-0^2)^{-1/2} = m$, the recursion delivers $u_3, u_5, u_7, \ldots$, and so Euler's result in $x(\alpha) = \sin(\alpha)$.

References

[1] P. J. Larcombe, *The 18th century Chinese discovery of the Catalan numbers*, Mathematical Spectrum, **32** (1999/2000), 5–7.

[2] P. J. Larcombe, *On Catalan numbers and expanding the sine function*, Bulletin of the I.C.A., **28** (2000), 39–47.

[3] P. J. Larcombe, *On a new formulation of Xinrong for the embedding of Catalan numbers in series forms of the sine function*, J. Combinatorial Mathematics and Combinatorial Computing, **42** (2002), 209–221.

[4] P. J. Larcombe, *On certain series expansions of the sine function containing embedded Catalan numbers: a complete analytic formulation*, J. Combinatorial Mathematics and Combinatorial Computing, **59** (2006), 3–16.

[5] P. J. Larcombe and D. R. French, *On expanding the sine function with Catalan numbers: a note on a role for hypergeometric functions*, J. Combinatorial Mathematics and Combinatorial Computing, **37** (2001), 65–74.

[6] J. Luo, *Antu Ming, the first inventor of Catalan numbers in the world*, Neimenggu Daxue Xuebao, **19** (1988), 239–245.

[7] M. Xinrong, *The general solution of Ming Antu's problem*, Acta Mathematica Sinica (English Series), **20** (2004), 157–162.

MSC2010: 30B10, 11K31, 40A05

School of Computing and Mathematics, University of Derby, Kedleston Road, Derby DE22 1GB, England, U.K.

E-mail address: p.j.larcombe@derby.ac.uk

Institute for Learning Enhancement and Innovation, University of Derby, Kedleston Road, Derby DE22 1GB, England, U.K.

E-mail address: S.Oneill@derby.ac.uk

BAE Systems Integrated System Technologies, Broad Oak, The Airport, Portsmouth PO3 5PQ, England, U.K.

E-mail address: Eric.Fennessey@baesystems.com