FIBONACCI NUMBERS CLOSE TO A POWER OF 2

SHANE CHERN AND ALVIN CUI

Abstract. In this paper, we find all Fibonacci numbers which are close to a power of 2.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. There is a rich history on the problem of finding Fibonacci numbers of a particular form. For example, Bugeaud, Mignotte, and Siksek [3] showed that the only Fibonacci perfect powers are 0, 1, 8, 144. Authors also studied Fibonacci numbers of the form $q^a y^t$ [2], $y^t \pm 1$ [1], etc. For more details, see **D26** of Guy's famous book *Unsolved Problems in Number Theory* [5].

We say that a number n is *close* to a positive number m, if it satisfies

$$|n - m| < \sqrt{m}.$$

In this paper we are interested in Fibonacci numbers which are close to a power of 2. More precisely, our main result is the following theorem.

Theorem 1.1. *There are only 8 Fibonacci numbers which are close to a power of* 2*. Namely, the solutions* $(F_n, 2^m)$ *of the inequality*

$$|F_n - 2^m| < 2^{m/2} \tag{1.1}$$

are $(1,2)$, $(2,2)$, $(3,2)$, $(3,4)$, $(5,4)$, $(8,8)$, $(13,16)$, *and* $(34,32)$.

2. Preliminaries

We first recall the Binet formula for Fibonacci numbers,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n \geq 0, \tag{2.1}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2 = -1/\alpha$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. It implies that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{2.2}$$

holds for all $n \geq 1$.

Then, we recall a lower bound for a linear form in logarithms which is given by Matveev [7].

Lemma 2.1. *Let* $\mathbb{K}$ *be a number field of degree* D *over* $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ *be positive reals of* $\mathbb{K}$, *and* $b_1, \ldots, b_t$ *be rational integers. Let*

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$\Lambda := 1 - \gamma_1^{b_1} \cdots \gamma_t^{b_t}.$$

Let $A_1, \ldots, A_t$ *be real numbers such that*

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$*, we have*

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t.$$

As usual, the *logarithmic height* of a d-degree algebraic number γ is defined as

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right),$$

with

$$f(X) := a_0 \prod_{i=1}^{d}\left(X - \gamma^{(i)}\right) \in \mathbb{Z}[X]$$

being the minimal primitive polynomial over the integers with positive leading coefficient a_0 and γ as a root.

At last, to reduce the upper bound which is generally too large, we need a variant of the Baker-Davenport Lemma, which is due to Dujella and Pethö [4]. Here, for a real number x, let $||x|| := \min\{|x - n| : n \in \mathbb{Z}\}$ denote the distance from x to the nearest integer.

Lemma 2.2. *Suppose that* M *is a positive integer, and* A*,* B *are positive reals with* $B > 1$*. Let* p/q *be the convergent of the continued fraction expansion of the irrational number* γ *such that* $q > 6M$*, and let* $\epsilon = ||\mu q|| - M||\gamma q||$*, where* μ *is a real number. If* $\epsilon > 0$*, then there is no solution of the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m *and* n *with*

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

Now, we are ready to prove our main result. The proof is somewhat motivated by Marques and Togbé [6].

3. Proof of Theorem 1.1

3.1. **The case** $m \leq 10$**.** In Table 1, we list the first 10 intervals of $S_m := (2^m - 2^{m/2}, 2^m + 2^{m/2})$, and find all the Fibonacci numbers in them. We get that the only Fibonacci numbers which are close to 2^m with $m \leq 10$ are $1, 2, 3, 5, 8, 13, 34$.

3.2. **The case** $m > 10$**.** By (2.1), we have

$$\left|F_n - \frac{\alpha^n}{\sqrt{5}}\right| = \frac{1}{\sqrt{5}\alpha^n}.$$

Combining it with (1.1), we get

$$\left|2^m - \frac{\alpha^n}{\sqrt{5}}\right| < 2^{m/2} + \frac{1}{\sqrt{5}\alpha^n},$$

Table 1

m	Integers in S_m	Fibonacci numbers in S_m
1	$1, 2, 3$	$1, 2, 3$
2	$3, 4, 5$	$3, 5$
3	$6, 7, 8, 9$	8
4	$13, 14, \ldots, 19$	13
5	$27, 28, \ldots, 37$	34
6	$57, 58, \ldots, 71$	
7	$117, 118, \ldots, 139$	
8	$241, 242, \ldots, 271$	
9	$490, 491, \ldots, 534$	
10	$993, 994, \ldots, 1055$	

which can be rewritten as

$$\left|1 - 2^{-m}\alpha^n\sqrt{5}^{-1}\right| < 2^{-m/2} + \frac{1}{2^m\alpha^n\sqrt{5}} < 2^{-m/2+1}. \tag{3.1}$$

In order to apply Lemma 2.1, we take $\gamma_1 = 2$, $\gamma_2 = \alpha$, and $\gamma_3 = \sqrt{5}$. For this choice, we have $D = 2$, $h(\gamma_2) = (\log\alpha)/2$, and $h(\gamma_3) = (\log 5)/2$. Thus, we can take $A_1 = 2\log 2$, $A_2 = \log\alpha$, and $A_3 = \log 5$. Also, according to (1.1) and (2.2), we have

$$2^m - 2^{m/2} < F_n < \alpha^{n-1},$$

which yields $m \le n$. Hence, we have $B = n$. It is easy to see that

$$\Lambda = 1 - 2^{-m}\alpha^n\sqrt{5}^{-1} \neq 0.$$

By Lemma 2.1, we get

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) \times (1 + \log n) \times 2\log 2 \times \log\alpha \times \log 5.$$

From (3.1), we have

$$\log|\Lambda| < (-m/2 + 1)\log 2.$$

Therefore, we get

$$m/2 - 1 < 1.6 \times 10^{12} \times (1 + \log n). \tag{3.2}$$

By (1.1) and (2.2), we have,

$$\alpha^{n-2} < F_n < 2^m + 2^{m/2} < \alpha^{-2} \cdot 2^{m+2},$$

which yields

$$n < ((m + 2)\log 2)/(\log\alpha). \tag{3.3}$$

Combining it with (3.2), and by a calculation in *Mathematica*, we obtain

$$m < 1.1 \times 10^{14} \quad \text{and} \quad n < 1.6 \times 10^{14}.$$

Now we are going to reduce the upper bounds of m and n. According to Bugeaud, Mignotte, and Siksek [3], no Fibonacci number equals 2^m when $m > 10$. Therefore, we discuss this case in two parts.

(I) $F_n > 2^m$. Noting that

$$\alpha^n/\sqrt{5} > F_n - 1 \geq 2^m,$$

we have

$$-m\log 2 + n\log\alpha - \log\sqrt{5} > 0.$$

Since $x < e^x - 1$, using (3.1) and (3.3), we get

$$\begin{aligned} 0 < -m\log 2 + n\log\alpha - \log\sqrt{5} &< 2^{-m/2+1} \\ &< 2^{-\frac{\log\alpha}{2\log 2}n+2} \\ &< 4\times 1.25^{-n}. \end{aligned} \tag{3.4}$$

By dividing by $\log 2$ on both sides above, (3.4) can be rewritten as

$$0 < n\frac{\log\alpha}{\log 2} - m - \frac{\log\sqrt{5}}{\log 2} < \frac{4}{\log 2}\times 1.25^{-n}. \tag{3.5}$$

To apply Lemma 2.2, we take $\gamma = (\log\alpha)/(\log 2)$, $\mu = (-\log\sqrt{5})/(\log 2)$, $A = 4/(\log 2)$, and $B = 1.25$. It is easy to see that γ is irrational. Let q_n be the denominator of the nth convergent of the continued fraction of γ. Taking $M = 1.6\times 10^{14}$, we have

$$q_{34} = 2683806884597620 > 6M,$$

and then $\epsilon = ||\mu q_{34}|| - M||\gamma q_{34}|| = 0.436226\ldots$. Hence there is no solution to inequality (3.5) (and then no solution to inequality (1.1)) for n in the range

$$\left[\left\lfloor\frac{\log(Aq_{34}/\epsilon)}{\log B}\right\rfloor + 1, M\right] \supset [171, 1.6\times 10^{14}].$$

Thus, $n < 171$.

(II) $F_n < 2^m$. Note that for negative x, we have

$$0 < -x < e^{-x} - 1 = e^{-x}\left|e^x - 1\right|.$$

Here, we take

$$x = -m\log 2 + n\log\alpha - \log\sqrt{5} < 0.$$

Note also that

$$|e^x - 1| < \frac{4}{1.25^n} < \frac{1}{2}.$$

Since x is negative, this shows that $e^x \in (1/2, 1)$, so that $e^{-x} < 2$. Now, we obtain

$$0 < m\frac{\log 2}{\log\alpha} - n + \frac{\log\sqrt{5}}{\log\alpha} < \frac{8}{\log\alpha}\times 1.25^{-n}.$$

Through a similar argument, we get $m < 174$ and $n < 254$.

3.3. **A calculation in *Mathematica*.** Let $x = (\log F_n)/(\log 2)$. Note that $x > 4$ since $n \geq m > 10$. Note also that

$$2^{x+1} - 2^{(x+1)/2} > 2^x = F_n.$$

Therefore, for the case $F_n > 2^m$, we have

$$\frac{\log F_n}{\log 2} < m < \frac{\log F_n}{\log 2} + 1.$$

So we only need to check whether F_n is in $(2^m, 2^m + 2^{m/2})$, where

$$m = \left\lfloor \frac{\log F_n}{\log 2} \right\rfloor + 1.$$

Through a calculation in *Mathematica*, we conclude that there is no such n. For the case $F_n < 2^m$, through a similar calculation, we deduce that no such n exists. This completes the proof.

4. Comments

If we replace the base 2 in Theorem 1.1 by an arbitrary positive integer $a \geq 2$, we can see that there are finitely many Fibonacci numbers which are close to a^m for each a, respectively. Indeed, the arguments give a relatively small upper bound of $n(a)$ (or $m(a)$) for small a.

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Department of Mathematics, Zhejiang University, Hangzhou, 310027, China
E-mail address: chenxiaohang92@gmail.com

Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh PA, 15213
E-mail address: alvin.cui@sv.cmu.edu