SOME POLYGONAL NUMBER SUMMATION FORMULAS

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Abstract. Various relationships involving the polygonal (or figurate) numbers are investigated. Several summation formulas for the general case as well as examples of specific types of polygonal numbers are obtained.

1. Introduction

The kth polygonal number of rank r is given by [2, 14, 21]

$$\mathcal{P}_k^{(r)} = \frac{k[(r-2)k-(r-4)]}{2}. \tag{1.1}$$

Examples will be obtained for sums of the following polygonal numbers (listed with their respective OEIS classification numbers [10]): pentagonal: A00326 [20], hexagonal: A000384 [17], heptagonal: A000566 [16], octagonal: A000567 [19], and nonagonal: A001106 [18]. The summation formulas for powers of positive integers will be needed and can be found in various sources [6, 9, 11].

Before presenting the summation formulas we note the following interesting observations about polygonal numbers.

Proposition 1.1. *The polygonal numbers satisfy the recurrence relation*

$$y_{k+2} - 2y_{k+1} + y_k = r-2.$$

Proof. Using the standard technique for solving recurrence relations with constant coefficients [13], with initial conditions, $y_1 = 1$ and $y_2 = r$ in $y_k = C_1 + C_2 k + Bk^2$ yields

$$y_k = 0 + \frac{1}{2}(4-r)k + \frac{1}{2}(r-2)k^2 = \frac{1}{2}k\left[(r-2)k-(r-4)\right]$$

which is (1.1). □

Proposition 1.2. *If $n \geq 3$ then $\sum\limits_{k=0}^{n} \binom{n}{k}(-1)^{n+k}\mathcal{P}_{n+k}^{(r)} = 0$.*

Proof. Using Euler's finite difference theorem, [6, 11],

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}k^j = \begin{cases} 0, & \text{if } 0 \leq j < n; \\ (-1)^n n!, & \text{if } j = n, \end{cases} \tag{1.2}$$

it follows that $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}\mathcal{P}_{n+k}^{(r)}$ is given by

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}\left[(r-2)(n^2+2nk+k^2)-(r-4)(n+k)\right],$$

which yields

$$\left[(r-2)n^2-(r-4)n\right]\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}k^0+[2(r-2)n-(r-4)]$$
$$\times\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}k^1+(r-2)\sum_{k=0}^{n}\binom{n}{k}(-1)^{n+k}k^2=0.$$

□

Additional formulas, identities, and relationships can be found in [4] and the references cited there.

We begin by presenting some general polygonal number summation formulas.

2. Summation Formulas for Polygonal Numbers

Using (1.1) and the special cases for sums of powers of integers [6, 9, 11], the sum of the first n polygonal numbers is given by

$$\sum_{k=1}^{n}\mathcal{P}_k^{(r)}=\frac{n(n+1)\left[(r-2)n+5-r\right)]}{6}. \tag{2.1}$$

The proof of (2.1) is routine but since it very simply illustrates how the proofs of the more complicated cases (which will not be presented due to their length) proceed, it is included here.

Proof.

$$\sum_{k=1}^{n}\mathcal{P}_k^{(r)}=\frac{r-2}{2}\sum_{k=0}^{n}k^2-\frac{r-4}{2}\sum_{k=0}^{n}k=\frac{n(n+1)[(r-2)(2n+1)-3(r-4)]}{12}$$
$$=\frac{n(n+1)[(r-2)n+5-r]}{6}.$$

□

Again, from (1.1) it follows that

$$(\mathcal{P}_k^{(r)})^2=\frac{(r-2)^2k^4-2(r-2)(r-4)k^3+(r-4)^2k^2}{4}.$$

Hence,

$$\sum_{k=1}^{n}(\mathcal{P}_k^{(r)})^2=\frac{n(n+1)}{60}\left[3(r-2)^3n^3-3(r-2)(r-7)n^2-(2r^2-3r-22)n+2(r^2-9r+19)\right]. \tag{2.2}$$

Next we consider the general case for powers of the polygonal numbers. The general formula for sums of powers of consecutive integers is given in terms of the Bernoulli numbers, B_i [1, 6, 7, 11, 12, 15]:

$$\sum_{k=1}^{n-1}k^m=\frac{(n+B)^{m+1}-B^{m+1}}{m+1}$$
$$=\frac{n^{m+1}+\binom{m+1}{1}n^mB+\cdots+\binom{m+1}{m}nB^m}{m+1}, \tag{2.3}$$

where the symbolic operator, B^i is replaced by the Bernoulli number, B_i [12]. Note that $B_1 = \frac{-1}{2}, B_{2i+1} = 0$, for $i \geq 1$; and $B_{2i} \in \left\{1, \frac{1}{6}, \frac{-1}{30}, \frac{1}{42}, \dots\right\}$ for $i \geq 0$.

To obtain the sum of the general powers of polygonal numbers, first note that

$$\begin{aligned}(\mathcal{P}_k^{(r)})^m &= \frac{k^m\left[(r-2)k-(r-4)\right]^m}{2^m} \\ &= \left(\frac{k}{2}\right)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j (r-2)^{m-j} (r-4)^j k^{m-j}.\end{aligned} \tag{2.4}$$

Thus,

$$\begin{aligned}\sum_{k=1}^{n} (\mathcal{P}_k^{(r)})^m &= \sum_{k=1}^{n} \left(\frac{k}{2}\right)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j (r-2)^{m-j} (r-4)^j k^{m-j} \\ &= \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j (r-2)^{m-j} (r-4)^j}{2m+1-j} \\ &\quad \times \sum_{i=0}^{2m+1-j} \binom{2m+1-j}{i} (n+1)^{2m+1-j-i} B_i.\end{aligned} \tag{2.5}$$

Next from (1.1)

$$\mathcal{P}_{mk+j}^{(r)} = \frac{(mk+j)\left[(r-2)(mk+j)-(r-4)\right]}{2}. \tag{2.6}$$

So it follows that

$$\begin{aligned}\sum_{k=1}^{n} \mathcal{P}_{mk+j}^{(r)} &= \frac{(r-2)m^2}{6} n^3 + \frac{(r-2)m^2 + 2(r-2)mj - (r-4)m}{4} n^2 \\ &\quad + \frac{(r-2)m^2 + 6(r-2)mj + 6(r-2)j^2 - 3(r-4)m - 6(r-4)j}{12} n.\end{aligned} \tag{2.7}$$

In the interest of brevity, next we let

$A = (r-2)^2$,
$B = 2(r-2)\left[(r-2)(M+N)-(r-4)\right]$,
$C = (r-2)^2(M^2+4MN+N^2) - 3(r-2)(r-4)(M+N) + (r-4)^2$,
$D = 2(r-2)^2 MN(M+N) - (r-2)(r-4)(M^2+4MN+N^2) + (r-4)^2(M+N)$, and
$E = (r-2)^2 M^2 N^2 - (r-2)(r-4)MN(M+N) + (r-4)^2 MN$.

Then,

$$\mathcal{P}_{k+M}^{(r)} \mathcal{P}_{k+N}^{(r)} = \frac{Ak^4 + Bk^3 + Ck^2 + Dk + E}{4}. \tag{2.8}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n}\mathcal{P}_{k+M}^{(r)}\mathcal{P}_{k+N}^{(r)} &= \sum_{k=1}^{n}\frac{Ak^4+Bk^3+Ck^2+Dk+E}{4} \\ &= \frac{12An^5+15(2A+B)n^4+10(2A+3B+2C)n^3}{240} \\ &\quad + \frac{15(B+2C+2D)n^2+2(-A+5C+15D+30E)n}{240}.\end{aligned} \tag{2.9}$$

We now present summation formulas for the pentagonal numbers.

3. Pentagonal Numbers

Setting $r=5$ in (1.1) the pentagonal numbers are obtained

$$\mathcal{P}_k^{(5)} = P_k = \frac{k(3k-1)}{2},\ \{1,5,12,22,35,51,\ldots\},$$

and from (2.1) the sum of the pentagonal numbers is found to be

$$\sum_{k=1}^{n} P_k = \frac{n^2(n+1)}{2},\ \{1,6,18,40,75,126,\ldots\}.$$

Since

$$P_k^2 = \frac{k^2(9k^2-6k+1)}{4},$$

it follows that

$$\sum_{k=1}^{n} P_k^2 = \frac{n(n+1)(27n^3+18n^2-13n-2)}{60},\ \{1,26,170,654,1879,4480,\ldots\}.$$

Also, (2.4) yields

$$P_k^m = \frac{k^m(3k-1)^m}{2^m} = \left(\frac{k}{2}\right)^m \sum_{j=0}^{m}\binom{m}{j}(-1)^j 3^{m-j}k^{m-j},$$

and so (2.5) yields

$$2^m\sum_{k=1}^{n} P_k^m = \sum_{k=1}^{n} k^m(3k-1)^m = \sum_{k=1}^{n}\sum_{j=0}^{m}\binom{m}{j}(-1)^j 3^{m-j}k^{2m-j}.$$

Cubing (1.1) with $r=5$ yields

$$P_k^3 = \frac{k^3(27k^3-27k^2+9k-1)}{8}.$$

So,

$$\sum_{k=1}^{n} P_k^3 = \frac{n(n+1)(135n^5+180n^4-117n^3-128n^2+58n+12)}{280}, \tag{3.1}$$
$$\{1,126,1854,12502,55377,188028,\ldots\}.$$

Using (2.5) is routine but cumbersome. So details of the proofs are omitted. However, to illustrate the process, an abbreviated outline of the derivation of (3.1) is provided.
Outline of Proof for (3.1)*:*

$$\begin{aligned}\sum_{k=1}^{n} P_k^3 &= \frac{1}{8}\sum_{j=0}^{3}\binom{3}{j}\frac{(-1)^j 3^{3-j}}{7-j}\sum_{i=0}^{7}\binom{7-j}{i}(n+1)^{7-j-i}B_i\\ &= \frac{27}{56}\sum_{i=0}^{7}\binom{7}{i}(n+1)^{7-i}B_i - \frac{9}{16}\sum_{i=0}^{6}\binom{6}{i}(n+1)^{6-i}B_i\\ &\quad + \frac{9}{40}\sum_{i=0}^{5}\binom{5}{i}(n+1)^{5-i}B_i - \frac{1}{32}\sum_{i=0}^{4}\binom{4}{i}(n+1)^{4-i}B_i,\end{aligned}$$

which yields (3.1). □

Next letting $m = 2$, $j = 0$, and $j = -1$ in (2.7) yields, respectively,

$$\sum_{k=1}^{n} P_{2k} = \frac{n(n+1)(4n+1)}{2},\ \{5, 27, 78, 170, 315, 525, \ldots\},$$

and

$$\sum_{k=1}^{n} P_{2k-1} = \frac{n(4n^2-n-1)}{2},\ \{1, 13, 48, 118, 235, 411, \ldots\}.$$

Some identities for P_{3k+j} were obtained in [8] but here (2.7) is applied. So if $m = 3$, $j = 0$, and $j = -1$ then

$$\sum_{k=1}^{n} P_{3k} = \frac{3n(n+1)(3n+1)}{2},\ \{12, 63, 180, 390, 720, 1197, \ldots\}$$

and

$$\sum_{k=1}^{n} P_{3k-1} = \frac{n(3n-1)(3n+2)}{2},\ \{5, 40, 132, 308, 595, 1020, \ldots\}.$$

Setting $r = 5$, $m = 3$, and $j = 2$ in (2.7) yields

$$\sum_{k=1}^{n} P_{3k-2} = \frac{n(9n^2-6n-1)}{2},\ \{1, 23, 93, 238, 485, 861, \ldots\}.$$

Again, in the interest of brevity, we let

$$\begin{aligned}A &= 18M + 18N - 6,\\ B &= 9M^2 + 36MN + 9N^2 - 9M + 1,\\ C &= 18M^2N + 18MN^2 - 3M^2 - 12MN - 3N^2 + M + N, \text{ and}\\ D &= 9M^2N^2 - 3M^2N - 3MN^2 + MN.\end{aligned}$$

Then

$$P_{k+M}P_{k+N} = \frac{9k^4 + Ak^3 + Bk^2 + Ck + D}{4}.$$

So,

$$\sum_{k=1}^{n} P_{k+M}P_{k+N} = \frac{9}{20}n^5 + \frac{18+A}{16}n^4 + \frac{18+3A+2B}{24}n^3 + \frac{A+2B+2C}{16}n^2 + \frac{-9+5B+15C+30D}{120}n.$$

Hence,

$$P_kP_{k+1} = \frac{k(k+1)(3k+2)(3k-1)}{4}$$

and

$$\sum_{k=1}^{n} P_kP_{k+1} = \frac{n(n+1)(54n^3+171n^2+109n-34)}{120},$$

$$\{5, 65, 329, 1099, 2884, 6454, \ldots\}.$$

We conclude with some further examples.

4. Summation Formulas for Additional Polygonal Numbers

Substituting the appropriate value of r in the general polygonal number formula (1.1) and using the various identities developed in Section 2, the following identities are obtained.

Hexagonal Numbers

$$\mathcal{P}_k^{(6)} = H_k = k(2k-1), \qquad \{1, 6, 15, 28, 45, 66, \ldots\}$$

$$\sum_{k=1}^{n} H_k = \frac{n(n+1)(4n-1)}{6}, \qquad \{1, 7, 22, 50, 95, 161, \ldots\}$$

$$\sum_{k=1}^{n} H_k^2 = \frac{n(n+1)(24n^3+6n^2-16n+1)}{30}, \qquad \{1, 37, 262, 1046, 3071, 7427, \ldots\}$$

$$\sum_{k=0}^{n} H_kH_{k+1} = \frac{n(n+1)(4n^3+11n^2+4n-4)}{5}, \qquad \{6, 96, 516, 1776, 4746, 10752, \ldots\}.$$

Heptagonal Numbers

$$\mathcal{P}_k^{(7)} = h_k = \frac{k(5k-3)}{2}, \qquad \{1, 7, 18, 34, 55, 81, \ldots\}$$

$$\sum_{k=1}^{n} h_k = \frac{n(n+1)(5n-2)}{6}, \qquad \{1, 8, 26, 60, 115, 196, \ldots\}$$

$$\sum_{k=1}^{n} h_k^2 = \frac{n(n+1)(15n^3-11n+2)}{12}, \qquad \{1, 50, 374, 1530, 4555, 11116, \ldots\}$$

$$\sum_{k=0}^{n} h_kh_{k+1} = \frac{n(n+1)(30n^3+75n^2+13n-34)}{24}, \qquad \{7, 133, 745, 2615, 7070, 16142, \ldots\}.$$

Octagonal Numbers

$$\mathcal{P}_k^{(8)} = O_k = k(3k-2), \qquad \{1, 8, 21, 40, 65, 96, \ldots\}$$

$$\sum_{k=1}^{n} O_k = \frac{n(n+1)(2n-1)}{2}, \qquad \{1, 9, 30, 70, 135, 231, \ldots\}$$

$$\sum_{k=1}^{n} O_k^2 = \frac{n(n+1)(54n^3-9n^2-41n+11)}{30}, \qquad \{1, 65, 506, 2106, 6331, 15547, \ldots\}$$

$$\sum_{k=0}^{n} O_k O_{k+1} = \frac{n(n+1)(27n^3+63n^2+2n-32)}{15}, \qquad \{8, 176, 1016, 3616, 9856, 22624, \ldots\}.$$

Nonagonal Numbers

$$\mathcal{P}_k^{(9)} = N_k = \frac{k(7k-5)}{2}, \qquad \{1, 9, 24, 46, 75, 111, \ldots\}$$

$$\sum_{k=1}^{n} N_k = \frac{n(n+1)(7n-4)}{6}, \qquad \{1, 10, 34, 80, 155, 266, \ldots\}$$

$$\sum_{k=1}^{n} N_k^2 = \frac{n(n+1)(147n^3-42n^2-113n+38)}{60}, \qquad \{1, 82, 658, 2774, 8399, 20720, \ldots\}$$

$$\sum_{k=0}^{n} N_k N_{k+1} = \frac{n(n+1)(98n^3+217n^2-17n-118)}{40}, \qquad \{9, 225, 1329, 4779, 13104, 30198, \ldots\}.$$

5. Concluding Remarks

More identities would undoubtedly arise using the above methods and the generating function for the polygonal numbers, $\mathcal{P}^{(r)}(x) = (x[(r-3)x+1])/(1-x)^3$ [21] . Also the case of infinite reciprocal sums has been addressed in general in [5] and specifically for some triangle number patterns in [3]. However the authors believe that specific examples of sums of reciprocals of various polygonal numbers as well as identities for finite sums are worth further investigation.

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