FACTORING CHEBYSHEV POLYNOMIALS

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Abstract. We provide an organizational structure for the irreducible factors of Chebyshev polynomials of the first and second kind. Several new proofs of known results are given and extensions to compositions are derived. Finally, the decomposition of the irreducible factors as linear combinations of Chebyshev polynomials is obtained and a connection to the cyclotomic polynomials is demonstrated.

1. Introduction

The Chebyshev polynomials of the first and second kind are defined by the formulas $T_n(\cos x) = \cos nx$ and $U_{n+1}(\cos x) = (\sin nx)/(\sin x)$. The factorization of these polynomials into irreducible factors has previously been discussed in [4, 6, 7, 10]. It is the goal of this paper to simplify and extend the results from these works. For related results, see [1, 2, 5, 8, 9].

We consider instead the normalized Chebyshev polynomials V_n and W_n defined by the properties that $V_n(2\cos x) = 2\cos nx$ and $W_n(2\cos x) = \frac{\sin nx}{\sin x}$. These differ from the standard Chebyshev polynomials T_n and U_n. In particular, $V_n(x) = 2T_n(x/2)$ and $W_n(x) = U_{n+1}(x/2)$. It is clear that factorization of V_n and W_n immediately gives factorizations of T_n and U_n. One key difference is that V_n and W_n are monic polynomials, which follows from the common recurrence relation $P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$ with initial conditions $V_0(x) = 2, V_1(x) = x, W_0(x) = 0$, and $W_1(x) = 1$. In particular, we have $V_2(x) = x^2 - 2, V_3(x) = x^3 - 3x, W_2(x) = x, W_3(x) = x^2 - 1$. We also note the identity $V_{mn} = V_n \circ V_m$ for $n, m \geq 1$.

We define the chebytomic polynomials $\psi_n(x)$ by setting $\psi_1(x) = x - 2$, $\psi_2(x) = x + 2$, and

$$\psi_n(x) = \prod_{\substack{\gcd(k,n)=1 \\ 0<k<n/2}} \left(x - 2\cos\frac{2\pi k}{n}\right), \tag{1.1}$$

for $n > 2$. For example, $\psi_3(x) = x + 1$ and $\psi_4(x) = x$. These polynomials are the same as the fibotomic polynomials, $Q_{2n}(x), Q^{\text{even}}_{2n+1}(x)$, and $Q^{\text{odd}}_{2n+1}(x)$ of Levy [6]. The current notation seems to be both cleaner and to allow better statements and proofs of results.

2. Basic Factorizations

Theorem 2.1. *The chebytomic polynomials are irreducible over $\mathbb{Q}$ and have integer coefficients.*

Proof. Let $\xi = \exp(\frac{2\pi i}{n})$ be the primitive nth root of unity. Let G be the Galois group of $\mathbb{Q}[\xi]$ over $\mathbb{Q}$. Each element of G takes ξ to ξ^k for some k with $(k, n) = 1$. Furthermore, G acts transitively on such ξ^k. Since $2\cos(\frac{2\pi k}{n}) = \xi^k + \xi^{-k}$, the roots of ψ_n are acted on transitively by G. Thus, ψ_n is irreducible over $\mathbb{Q}$ and has rational coefficients. Since the coefficients are also algebraic integers, they must be integers. □

In particular, ψ_n is the characteristic polynomial of the algebraic integer $2\cos\frac{2\pi}{n}$.

For notational convenience, let $e_n = 1$ if n is even and $e_n = 0$ if n is odd. We start with a factorization of $V_n - 2$.

Proposition 2.2. *The polynomial $V_n - 2$ factors as follows:*

$$V_n - 2 = \psi_1 \psi_2^{e_n} \prod_{\substack{k|n \\ k\neq 1,2}} \psi_k^2. \tag{2.1}$$

Proof. The roots of $V_n - 2$ are exactly $x_k = 2\cos(\frac{2\pi k}{n})$ for $0 \le k \le n$. All roots are double roots except $x_0 = 2$ and, in the case n is even, $x_{n/2} = -2$. Thus, the two sides of the claimed equality have the same roots with the same multiplicities. Since both sides are also monic polynomials, they are equal. □

Proposition 2.3. *The polynomial $V_n + 2$ factors as follows:*

$$V_n + 2 = \psi_2^{1-e_n} \prod_{\substack{k|2n \\ k\nmid n \\ k\neq 2}} \psi_k^2. \tag{2.2}$$

In particular, for m odd, we have

$$V_m + 2 = \psi_2 \prod_{\substack{k|m \\ k\neq 1}} \psi_{2k}^2, \tag{2.3}$$

and for $n \ge 1$ and m odd, we have

$$V_{2^n m} + 2 = \prod_{k|m} \psi_{2^{n+1}k}^2. \tag{2.4}$$

Proof. Use the fact that $V_n + 2 = (V_{2n} - 2)/(V_n - 2)$ and the previous result. □

This, in turn, gives us the decomposition of V_n into irreducible factors.

Theorem 2.4. *We have, for m odd, and $n \ge 0$,*

$$V_{2^n m} = \prod_{k|m} \psi_{2^{n+2}k}. \tag{2.5}$$

Proof. We have

$$V_{2^n m}^2 = V_{2^{n+1}m} + 2 = \prod_{\substack{k|2^{n+2}m \\ k\nmid 2^{n+1}m}} \psi_k^2 = \prod_{k|m} \psi_{2^{n+2}k}^2. \tag{2.6}$$

Since both sides of the proposed factorization are monic polynomials with the same square, they are equal. □

Corollary 2.5. *If m_1 and m_2 are odd, then $\gcd(V_{2^n m_1}, V_{2^n m_2}) = V_{2^n \gcd(m_1,m_2)}$. If $n_1 \neq n_2$, then $V_{2^{n_1} m_1}$ and $V_{2^{n_2} m_2}$ are relatively prime.*

This is an alternative statement of a result from [7].

We collect a few more basic factorizations in the next result. Some of these factorizations are to be found in [6] and [3].

Theorem 2.6. *We have the following factorizations of polynomials.*

a)

$$W_n = \prod_{\substack{k|2n \\ k\neq 1,2}} \psi_k. \tag{2.7}$$

b)

$$V_{n+1} + V_n = \prod_{k|2n+1} \psi_{2k}. \tag{2.8}$$

c)

$$V_{n+1} - V_n = \prod_{k|2n+1} \psi_k. \tag{2.9}$$

d)

$$W_{n+1} - W_n = \prod_{\substack{k|2n+1 \\ k\neq 1}} \psi_{2k}. \tag{2.10}$$

e)

$$W_{n+1} + W_n = \prod_{\substack{k|2n+1 \\ k\neq 1}} \psi_k. \tag{2.11}$$

f)

$$V_{n+1} - V_{n-1} = \prod_{k|2n} \psi_k. \tag{2.12}$$

Proof. From the Pythagorean identity, $V_n^2(x) + W_n^2(x)(4 - x^2) = 4$, we obtain $W_n^2 = (V_n^2 - 4)/(\psi_1\psi_2) = (V_{2n} - 2)/(\psi_1\psi_2)$. The factorization of $V_{2n} - 2$ above shows that both sides of the first identity have the same square. Since they are also monic polynomials, they are equal.

For the other factorizations, use the identities

$$\begin{aligned}
(V_{n+1} + V_n)^2 &= (V_{2n+1} + 2)\psi_2, \\
(V_{n+1} - V_n)^2 &= (V_{2n+1} - 2)\psi_1, \\
(W_{n+1} - W_n)^2\psi_2 &= V_{2n+1} + 2, \\
(W_{n+1} + W_n)^2\psi_1 &= V_{2n+1} - 2, \\
(V_{n+1} - V_{n-1})^2 &= (V_{2n} - 2)\psi_1\psi_2,
\end{aligned}$$

all of which follow easily from corresponding trigonometric identities. □

We point out that all of these can be used together with the Möbius inversion formula to obtain ψ_n for various n. We will find more efficient methods soon. However, a couple of immediate results should be noted, both of which are previously known, see [7].

Corollary 2.7. *We have that $V_{2^n} = \psi_{2^{n+2}}$ is irreducible for each n. These are the only V_n which are irreducible.*

For example, $\psi_8(x) = x^2 - 2$ and $\psi_{16}(x) = x^4 - 4x^2 + 2$.

Corollary 2.8. *The function $V_n(x)/x$ is an irreducible polynomial if and only if n is an odd prime. For odd prime p, we have $V_p(x)/x = \psi_{4p}(x)$.*

This follows from the factorization of V_n and the fact that $\psi_4(x) = x$. For example, $\psi_{12}(x) = V_3(x)/x = x^2 - 3$. This result is used in [8] to obtain a factorization test.

The following appears to be new.

Corollary 2.9. *If p is an odd prime, then $\psi_p = W_{\frac{p+1}{2}} + W_{\frac{p-1}{2}}$ and $\psi_{2p} = W_{\frac{p+1}{2}} - W_{\frac{p-1}{2}}$. Hence both expressions are irreducible. Furthermore, $W_{n+1} \pm W_n$ is irreducible if and only if $n = \frac{p-1}{2}$ for p an odd prime.*

For example, $\psi_3(x) = x + 1$, $\psi_5(x) = x^2 + x - 1$, $\psi_6(x) = x - 1$, $\psi_7(x) = x^3 + x^2 - 2x - 1$, $\psi_{10}(x) = x^2 - x - 1$, $\psi_{11}(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, $\psi_{13}(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1$, and $\psi_{14}(x) = x^3 - x^2 - 2x + 1$.

3. Factoring Compositions

Theorem 3.1. *If $n \geq 3$ is odd and $\gcd(m, n) = 1$, then $\psi_n \circ V_m = \prod_{k|m} \psi_{nk}$.*

Proof. In fact, for n odd and $\gcd(m, n) = 1$, we have

$$\begin{aligned} V_{nm} - 2 &= \psi_1 \psi_2^{e_m} \prod_{\substack{k|mn \\ k \neq 1,2}} \psi_k^2 \\ &= \psi_1 \psi_2^{e_m} \left(\prod_{\substack{k|m \\ k \neq 1,2}} \psi_k^2 \right) \prod_{\substack{k|n \\ k \neq 1}} \left(\prod_{\ell|m} \psi_{k\ell}^2 \right). \end{aligned}$$

Alternatively, we have, noting $\psi_1(x) = x - 2$,

$$\begin{aligned} V_{nm} - 2 &= V_n \circ V_m - 2 \\ &= (\psi_1 \circ V_m) \prod_{\substack{k|n \\ k \neq 1}} \psi_k^2 \circ V_m \\ &= \psi_1 \psi_2^{e_m} \left(\prod_{\substack{k|m \\ k \neq 1,2}} \psi_k^2 \right) \prod_{\substack{k|n \\ k \neq 1}} \psi_k^2 \circ V_m. \end{aligned}$$

Comparing these two expressions, using Möbius inversion, and noting that $\psi_k \circ V_m$ is a monic polynomial gives the result. □

The following reduces the computation of ψ_n to the case where n is square-free.

Theorem 3.2. *Suppose that n is not a power of 2 and that $n = p_1^{n_1} \cdots p_k^{n_k}$ is the factorization of n into primes with $n_j \neq 0$ for all j. Let $m = p_1 \cdots p_k$ be the square-free part of n. Then $\psi_n = \psi_m \circ V_{n/m}$.*

This follows by repeated use of the following lemmas.

Lemma 3.3. *If p is an odd prime, then $\psi_{p^{n+1}} = \psi_p \circ V_{p^n}$ and $\psi_{2p^{n+1}} = \psi_{2p} \circ V_{p^n}$.*

Proof. We have, from Proposition 2.2, $\psi_p^2 = \frac{V_p - 2}{\psi_1}$ and $\psi_{p^{n+1}}^2 = \frac{V_{p^{n+1}} - 2}{V_{p^n} - 2} = \frac{V_p \circ V_{p^n} - 2}{\psi_1 \circ V_{p^n}} = \psi_p^2 \circ V_{p^n}$. We get the other expression from the factorization of $V_{p^{n+1}} + 2$ in the same way. □

In particular, $\psi_9(x) = V_3(x) + 1 = x^3 - 3x + 1$.

Lemma 3.4. *Let $n \geq 3$ be odd, p a prime with $p \nmid n$. Then*

a) $\psi_{np} = \frac{\psi_n \circ V_p}{\psi_n}$.
b) $\psi_n \circ V_{p^m} = \prod_{k=0}^{m} \psi_{np^k}$.
c) $\psi_{np^{m+1}} = \psi_{np} \circ V_{p^m}$.
d) If, in addition, p is odd, $\psi_{2np^{m+1}} = \psi_{2np} \circ V_{p^m}$.

Proof. The first two statements are direct applications of Theorem 3.1. For the third, notice that

$$\psi_{np^{m+1}} = \frac{\psi_n \circ V_{p^{m+1}}}{\psi_n \circ V_{p^m}} = \frac{\psi_n \circ V_p \circ V_{p^m}}{\psi_n \circ V_{p^m}}, \tag{3.1}$$

while

$$\psi_{np} = \frac{\psi_n \circ V_p}{\psi_n}. \tag{3.2}$$

Again by the theorem, and noting that $V_m \circ V_n = V_{mn} = V_n \circ V_m$, we have

$$\psi_{2np^{m+1}} = \frac{\psi_{np^{m+1}} \circ V_2}{\psi_{np^{m+1}}} = \frac{\psi_{np} \circ V_{p^m} \circ V_2}{\psi_{np} \circ V_{p^m}} = \frac{\psi_{np} \circ V_2 \circ V_{p^m}}{\psi_{np} \circ V_{p^m}}, \tag{3.3}$$

and

$$\psi_{2np} = \frac{\psi_{np} \circ V_2}{\psi_{np}}, \tag{3.4}$$

which gives the last result. □

As an example, if $p \neq 3$ is an odd prime, then $\psi_{3p}(x) = (V_p(x) + 1)/(x + 1)$. So, $\psi_{15}(x) = x^4 - x^3 - 4x^2 + 4x + 1$. We will return to this example below. This completes the evaluation of ψ_n for $n \leq 16$.

Theorem 3.5. *If $n > 2$, then*

$$\psi_n \circ V_m = \prod_{\substack{k|m \\ \gcd(k,n)=1}} \psi_{\frac{mn}{k}}. \tag{3.5}$$

Proof. Write $n = p_1^{n_1} \cdots p_k^{n_k}$ for the factorization into primes with $n_j > 0$ and write $m = p_1^{m_1} \cdots p_k^{m_k} \cdot a$ where $m_j \geq 0$ for all j and $\gcd(n, a) = 1$. Then, $\psi_n \circ V_m = \psi_n \circ V_{\frac{m}{a}} \circ V_a = \psi_{\frac{nm}{a}} \circ V_a$. □

We note that $\psi_1 \circ V_n = V_n - 2$ and $\psi_2 \circ V_n = V_n + 2$ have already been factored above. With $n = 3$ and $n = 6$, we obtain factorizations of $V_n + 1$ and $V_n - 1$, respectively.

4. Additive Properties

Since V_n, $n \geq 1$ is a monic polynomial of degree n, it is clear that every integer polynomial can be written as a linear combination of the V_n with integer coefficients plus a constant term. The question then arises how ψ_n can be written in this way. If $n \geq 8$ is a power of 2, we have that $\psi_n = V_{\frac{n}{4}}$, so this case is trivial. We explore a couple of other special cases before giving the general result.

Proposition 4.1. *Suppose that p is an odd prime. Then*

$$\psi_p = 1 + \sum_{n=1}^{(p-1)/2} V_n.$$

Proof. Consider the sequence of trigonometrical identities

$$\begin{aligned}\psi_p(2\cos x) &= W_{\frac{p-1}{2}}(2\cos x) + W_{\frac{p-1}{2}}(2\cos x)\\ &= \frac{\sin\frac{px}{2}}{\sin\frac{x}{2}}\\ &= 1 + 2\sum_{n=1}^{(p-1)/2}\cos(nx)\\ &= 1 + \sum_{n=1}^{(p-1)/2} V_n(2\cos x).\end{aligned}$$

The claimed equality follows. □

Proposition 4.2. *Let $p \neq 3$ be an odd prime. Let $r_p = 1$ if $p \equiv 1 \pmod 3$, and let $r_p = V_1 - 1$ if $p \equiv 2 \pmod 3$. Then*

$$\psi_{3p} = r_p + \sum_{k<(p-2)/3} \left(V_{p-1-3k} - V_{p-2-3k}\right).$$

Proof. First notice that $(x+1)\psi_{3p}(x) = \psi_3(x)\psi_{3p}(x) = V_p(x) + 1$.

$$\begin{aligned}V_n(x) + 1 &= xV_{n-1}(x) - V_{n-2}(x) + 1\\ &= (x+1)V_{n-1}(x) - V_{n-1}(x) - V_{n-2}(x) + 1\\ &= (x+1)V_{n-1}(x) - xV_{n-2}(x) - V_{n-2}(x) + V_{n-3}(x) + 1\\ &= (x+1)(V_{n-1}(x) - V_{n-2}(x)) + V_{n-3}(x) + 1.\end{aligned}$$

Now proceed inductively until either $V_2(x) + 1 = (x+1)(V_1(x) - 1)$ or $V_1(x) + 1 = x + 1$ is reached. □

Of course, the previous technique gives a factorization of $V_n + 1$ for any n not divisible by 3. But it is only in the case of n prime that the factor other than $x + 1$ is irreducible.

We now give the general decomposition of ψ_n in terms of the V_k.

Theorem 4.3. *Let $n > 2$ and write the cyclotomic polynomial $\Phi_n(x) = \sum a_k x^k$ where k runs from 0 to $d = \phi(n)$ and $a_{d-k} = a_k$. Then,*

$$\psi_n = a_{d/2} + \sum_{k=1}^{d/2} a_{\frac{d-2k}{2}} \cdot V_k. \tag{4.1}$$

Proof. Let $f(x)$ be the polynomial of the right side of this equation and $\xi = \exp(\frac{2\pi i}{n})$, so Φ_n is the characteristic polynomial of ξ. Then,

$$\begin{aligned}
0 &= \Phi_n(\xi) \cdot \xi^{-d/2} \\
&= \sum_{k=0}^{d} a_k \xi^{(2k-d)/2} \\
&= a_{d/2} + \sum_{k=0}^{d/2-1} a_k \left(\xi^{(2k-d)/2} + \xi^{(d-2k)/2}\right) \\
&= a_{d/2} + \sum_{k=1}^{d/2} a_{\frac{d-2k}{2}} \cdot \left(\xi^k + \xi^{-k}\right) \\
&= a_{d/2} + \sum_{k=1}^{d/2} a_{\frac{d-2k}{2}} \cdot 2\cos\left(\frac{2\pi k}{n}\right) \\
&= a_{d/2} + \sum_{k=1}^{d/2} a_{\frac{d-2k}{2}} \cdot V_k\left(2\cos\frac{2\pi}{n}\right) \\
&= f\left(2\cos\frac{2\pi}{n}\right).
\end{aligned}$$

Hence, $2\cos\left(\frac{2\pi}{n}\right)$ is a root of the monic ($a_0 = 1$) integer polynomial $f(x)$. But ψ_n is the characteristic polynomial of this root, has the same degree and is also monic. Hence, $f(x) = \psi_n$. □

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