FIXED-TERM ZECKENDORF REPRESENTATIONS

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Abstract. We consider here an aspect of Zeckendorf representations of integers. In particular, we obtain the structure of the set of positive integers such that each element of this set contains F_k in its Zeckendorf representation.

1. Introduction

In order to set the scene, by way of an example we let $\mathcal{D}$ denote the infinite set of all positive integers such that, when expressed in decimal notation, each has a '3' appearing in its 'tens' column. We term this a *fixed-term* decimal representation. Note that

$$\mathcal{D} = \{30, 31, 32, \dots, 39, 130, 131, 132, \dots, 139, 230, 231, 232, \dots, 239, 330, \dots\},$$

which may in fact be written somewhat more succinctly as follows:

$$\mathcal{D} = \{100n + 30 + j : 0 \leq j \leq 9, n \geq 0\}.$$

It is of course possible to obtain similar fixed-term representations of integers in binary, ternary, and so on.

In this paper we consider the corresponding situation for the Zeckendorf representation of integers. In particular, we study here the structure of the set of positive integers $Z(k)$ such that each of the integers in this set contains F_k in its Zeckendorf representation for some fixed $k \geq 2$. We also extend this idea by allowing more than one term to be fixed.

2. Some Initial Definitions and Results

Zeckendorf's Theorem [8] states that every $n \in \mathbb{N}$ has a unique representation as the sum of distinct Fibonacci numbers that does not include any consecutive Fibonacci numbers. Somewhat more formally, for any $n \in \mathbb{N}$ there exists an increasing sequence of positive integers of length $k \in \mathbb{N}$, $(c_1, c_2, \dots, c_k)$ say, such that $c_1 \geq 2$, $c_i \geq c_{i-1} + 2$ for $i = 2, 3, \dots, k$, and

$$n = \sum_{i=1}^{k} F_{c_i}.$$

Relatively straightforward proofs of this result are given in [1, 7].

As will be seen, there is an intimate connection between the elements of $Z(k)$ and a mathematical object known as the golden string [4, 5]. We will use known properties of the golden string in order to reveal the structure of the set $Z(k)$.

Definition 2.1. *Let X and Y be finite strings of symbols. We use $X : Y$ to denote the concatenation of X and Y.*

Definition 2.2. *The golden string,*

$$S = BABBABABBABBABABBABAB\dots,$$

is an infinite string that may be obtained recursively as follows. We start with the strings $S_1 = B$ and $S_2 = BA$. In order to obtain S_3 we concatenate S_2 and S_1 as follows:

$$S_3 = S_2 : S_1 = BAB.$$

Next,

$$S_4 = S_3 : S_2 = BABBA,$$

and so on. In general $S_k = S_{k-1} : S_{k-2}$ for $k \geq 3$, and it is clear that S_k contains F_{k+1} letters, comprising F_{k-1} A's and F_k B's, noting that $F_0 = 0$.

The following three lemmas will be used in Section 3, each of which is easily proved by induction.

Lemma 2.3. *For any $j \geq 1$, the (F_{2j})th character of S is B and the (F_{2j+1})th character of S is A.*

Lemma 2.4. *Let $m, n \in \mathbb{N}$. If F_j and F_k are the largest summands in the Zeckendorf representations of m and n, respectively, then $j > k$ implies that $m > n$.*

Lemma 2.5. *For $n \geq 1$,*

$$F_{k+2n+1} - (F_{k+2n} + F_{k+2n-2} + \cdots + F_{k+2}) = F_{k+1}$$

and

$$F_{k+2(n+1)} - (F_{k+2n+1} + F_{k+2n-1} + \cdots + F_{k+3}) = F_{k+2}.$$

3. Fixed-Term Representations

Let $\mathcal{X}_k$ denote the set of all positive integers such that each has F_k as the smallest summand in its Zeckendorf representation. Next, let $\mathcal{Q}_k = (q(1), q(2), q(3), \ldots)$ be the strictly increasing infinite sequence that results on arranging the elements of $\mathcal{X}_k$ into ascending numerical order. If we replace each term $q(i)$ in $\mathcal{Q}_k$ with an ordered list of the summands in its Zeckendorf representation, we obtain Table 1, in which the rth row contains the list of summands of the rth smallest element from $\mathcal{X}_k$.

Lemma 3.1. *For $j \geq 2$, the rows of Table 1 for which F_{k+j} is the largest summand are those numbered from $F_j + 1$ to F_{j+1} inclusive.*

Proof. As is easily checked, the statement of the lemma is true for $j = 2$ and $j = 3$. Now assume that it is true for all j such that $2 \leq j \leq m$ for some $m \geq 3$. By the inductive hypothesis, bearing in mind Lemma 2.4, the total number of rows for which the largest summand is no greater than F_{k+m-1} is

$$1 + (F_3 - F_2) + (F_4 - F_3) + \cdots + (F_m - F_{m-1}) = F_m.$$

Note further that, since no consecutive Fibonacci numbers may appear in any Zeckendorf representation, the number of rows in Table 1 for which the largest summand is no greater than F_{k+m-1} is the same as the number of rows for which F_{k+m+1} is the largest summand. It is thus the case that F_m gives the number of rows possessing F_{k+m+1} as their largest summand.

Finally, also by way of the inductive hypothesis, the rows for which F_{k+m} is the largest summand are those numbered from $F_m + 1$ to F_{m+1} inclusive. It follows that the rows having F_{k+m+1} as the largest summand are those numbered from $F_{m+1} + 1$ to $F_{m+1} + F_m = F_{m+2}$ inclusive, as required. ☐

F_k						
F_k	F_{k+2}					
F_k		F_{k+3}				
F_k			F_{k+4}			
F_k	F_{k+2}		F_{k+4}			
F_k				F_{k+5}		
F_k	F_{k+2}			F_{k+5}		
F_k		F_{k+3}		F_{k+5}		
F_k					F_{k+6}	
F_k	F_{k+2}				F_{k+6}	
F_k		F_{k+3}			F_{k+6}	
F_k			F_{k+4}		F_{k+6}	
F_k	F_{k+2}		F_{k+4}		F_{k+6}	
F_k						F_{k+7}
			$\vdots$			

Table 1. The Zeckendorf representations, in numerical order, of the positive integers having F_k as their smallest summand.

Lemma 3.2. *For $j \geq 1$,*

$$q(j+1) - q(j) = \begin{cases} F_{k+1}, & \text{if } A \text{ is the } j\text{th character of } S; \\ F_{k+2}, & \text{if } B \text{ is the } j\text{th character of } S. \end{cases} \tag{3.1}$$

Proof. It is straightforward to check that the statement of the lemma is true for j such that $1 \leq j \leq F_4 - 1$. Now assume that it is true for $1 \leq j \leq F_m - 1$ for some $m \geq 4$. From Lemma 3.1 we know that the first F_{m-1} rows of Table 1 are those for which the largest summand is no greater than F_{k+m-2}, and furthermore that the rows for which F_{k+m} is the largest summand are those numbered from $F_m + 1$ to F_{m+1} inclusive. The ordering of the rows, in conjunction with these results, then implies that

$$q\left(i + F_m\right) = q(i) + F_{k+m}$$

for $i = 1, 2, \ldots, F_{m-1}$.

We now have, for $i = 1, 2, \ldots, F_{m-1} - 1$,

$$\begin{aligned} q\left(i + 1 + F_m\right) - q\left(i + F_m\right) &= \left(q(i+1) + F_{k+m}\right) - \left(q(i) + F_{k+m}\right) \\ &= q(i+1) - q(i). \end{aligned}$$

By way of the construction of S, the substring comprising its first F_{m-1} characters is identical to the substring of its characters numbered from $F_m + 1$ to F_{m+1} inclusive. From this it follows that (3.1) is satisfied when $F_m + 1 \leq j \leq F_{m+1} - 1$.

We now just need to consider the case $q\left(F_m + 1\right) - q\left(F_m\right)$. On using Lemma 2.5 it may be seen that if m is even then

$$q\left(F_m + 1\right) - q\left(F_m\right) = F_{k+m} - \left(F_{k+m-1} + F_{k+m-3} + \cdots + F_{k+3}\right) = F_{k+2},$$

while if m is odd it is the case that

$$q\left(F_m + 1\right) - q\left(F_m\right) = F_{k+m} - \left(F_{k+m-1} + F_{k+m-3} + \cdots + F_{k+2}\right) = F_{k+1}.$$

From this, in conjunction with Lemma 2.3, it follows that (3.1) is true for all $1 \le j \le F_{m+1}-1$, thereby completing the proof of the lemma. □

We will need to make use of a further result associated with the golden string. The following appears in [6] and is proved in [4].

Lemma 3.3. *The number of B's amongst the first n characters of S is given by*

$$\left\lfloor \frac{n+1}{\phi} \right\rfloor,$$

where

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the golden ratio and $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

We are now in a position to be able to specify the structure of $Z(k)$, the set of all integers such that, for some fixed $k \ge 2$, each contains F_k in its Zeckendorf representation.

Theorem 3.4. *For $k \ge 2$, the set of all positive integers having the summand F_k in their Zeckendorf representation is given by*

$$Z(k) = \left\{ F_k \left\lfloor \frac{n+\phi^2}{\phi} \right\rfloor + nF_{k+1} + j : 0 \le j \le F_{k-1}-1, n \ge 0 \right\}.$$

Proof. From Lemma 3.2 we know that $\mathcal{X}_k$ is given by

$$\{F_k + a(n)F_{k+1} + b(n)F_{k+2} : n \in \mathbb{N}\},$$

where $a(n)$ and $b(n)$ denote the number of A's and B's, respectively, amongst the first n characters in the golden string. On using Lemma 3.3 it follows that

$$\begin{aligned} F_k + a(n)F_{k+1} + b(n)F_{k+2} &= F_k + \left(n - \left\lfloor \frac{n+1}{\phi} \right\rfloor\right) F_{k+1} + \left\lfloor \frac{n+1}{\phi} \right\rfloor F_{k+2} \\ &= F_k + \left\lfloor \frac{n+1}{\phi} \right\rfloor (F_{k+2} - F_{k+1}) + nF_{k+1} \\ &= F_k \left(1 + \left\lfloor \frac{n+1}{\phi} \right\rfloor\right) + nF_{k+1} \\ &= F_k \left\lfloor \frac{n+\phi^2}{\phi} \right\rfloor + nF_{k+1}, \end{aligned}$$

where, for the purposes of simplification in the final step, we have used the fact that $\phi^2 = \phi+1$.

Finally, the elements of the set $\{F_2, F_3, \ldots, F_{k-2}\}$ can be used to obtain the Zeckendorf representations of integers for which the largest summand is no greater than F_{k-2}. Such representations, which generate all the integers from 1 to $F_{k-1}-1$ inclusive, may be 'appended' to any Zeckendorf representation having F_k as its smallest summand, giving rise to another Zeckendorf representation. This completes the proof of the theorem. □

For example,

$$Z(5) = \{5, 6, 7, 18, 19, 20, 26, 27, 28, 39, 40, 41, 52, 53, 54, 60, 61, 62, \ldots\}.$$

Similarly, with $Z(k, k+2)$ denoting the set of all integers such that, for some fixed $k \geq 2$, each contains both F_k and F_{k+2} in its Zeckendorf representation, we have

$$Z(k, k+2) = \left\{ F_{k+2} \left\lfloor \frac{n + \phi^2}{\phi} \right\rfloor + nF_{k+3} + F_k + j : 0 \leq j \leq F_{k-1} - 1, n \geq 0 \right\},$$

and then, extending the notation in an obvious way,

$$Z(k, k+3) = \left\{ F_{k+3} \left\lfloor \frac{n + \phi^2}{\phi} \right\rfloor + nF_{k+4} + F_k + j : 0 \leq j \leq F_{k-1} - 1, n \geq 0 \right\}.$$

However, the structure of $Z(k, k+4)$ is somewhat different since we may now use F_{k+2} in the Zeckendorf representation. It is in fact given by

$$\left\{ F_{k+4} \left\lfloor \frac{n + \phi^2}{\phi} \right\rfloor + nF_{k+5} + F_k + j + mF_{k+2} : 0 \leq j \leq F_{k-1} - 1, n \geq 0, 0 \leq m \leq 1 \right\}.$$

We have, for example,

$$Z(5, 9) = \{39, 40, 41, 52, 53, 54, 128, 129, 130, 141, 142, 143, 183, 184, 185, \ldots\}.$$

Note that $Z(k, k+2)$, $Z(k, k+3)$ and $Z(k, k+4)$ are each subsets of $Z(k)$.

Readers interested in exploring Fibonacci representations of integers yet further might like to refer to two classic articles in this regard [2, 3].

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